Entrance and sojourn times for Markov chains. Application to (L, R)-random walks

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Abstract

In this paper, we provide a methodology for computing the probability distribution of sojourn times for a wide class of Markov chains. Our methodology consists in writing out linear systems and matrix equations for generating functions involving relations with entrance times. We apply the developed methodology to some classes of random walks with bounded integer-valued jumps.

Keywords: Sojourn time; Entrance time; (L, R)-random walks; Generating functions; Matrix equations.
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1 Introduction

We first introduce some settings: we denote by $\mathbb{Z}$ the set of all integers, by $\mathbb{Z}^+$ that of positive integers: $\mathbb{Z}^+ = \mathbb{Z} \cap (0, +\infty) = \{1, 2, \dots\}$, by $\mathbb{Z}^-$ that of negative integers: $\mathbb{Z}^- = \mathbb{Z} \cap (-\infty, 0) = \{\dots, -2, -1\}$, and by $\mathbb{Z}^\dagger$ that of non-negative integers: $\mathbb{Z}^\dagger = \mathbb{Z}^+ \cup \{0\} = \{0, 1, 2, \dots\}$. We have the partition $\mathbb{Z} = \mathbb{Z}^+ \cup \{0\} \cup \mathbb{Z}^-$. These settings will be related to space variables. We introduce other settings for time variables: $\mathbb{N}$ is the set of non negative integers: $\mathbb{N} = \{0, 1, 2, \dots\}$ and $\mathbb{N}^*$ is that of positive integers: $\mathbb{N}^* = \{1, 2, \dots\}$.

While nearest neighbour random walk on $\mathbb{Z}$ has been extensively studied, random walk with several neighbours seems to be less considered. In this paper, we consider this class of random walks, i.e., those evolving on $\mathbb{Z}$ with jumps of size not greater than R and not less than $-L$, for fixed positive integers L, R, (the so-called (L, R)-random walks). More specifically, we are interested in the time spent by the random walk in some subset of $\mathbb{Z}$, e.g., $\mathbb{Z}^\dagger$ up to some fixed time.

1.1 Nearest neighbour random walk

Let us detail the most well-known example of random walk on $\mathbb{Z}$: let $(S_m)_{m\in\mathbb{N}}$ be the classical symmetric Bernoulli random walk defined on $\mathbb{Z}$. The probability distribution of the sojourn time of the walk $(S_m)_{m\in\mathbb{N}}$ in $\mathbb{Z}^\dagger$ up to a fixed time $n \in \mathbb{N}^*$,

$$T_n = \#\{m \in \{0, \dots, n\} : S_m \geq 0\} = \sum_{m=0}^{n} \mathbb{1}_{\mathbb{Z}^\dagger}(S_m),$$

is well-known. A classical way for calculating it consists in using generating functions; see, e.g., [5, Chap. III, §4] or [18, Chap. 8, §11] for the case of the nearest neighbour random walk on $\mathbb{Z}$, and [5, Chap. XIV, §8] where the case of random walk with general integer-valued jumps is mentioned. The methodology consists in writing out equations for the generating function of the family of numbers $\mathbb{P}\{T_n = m\}$, $m, n \in \mathbb{N}$. A representation for the probability distribution of T_n can be derived with the aid of Sparre Andersen's theorem (see [19, 20] and, e.g., [21, Chap. IV, §20]). Moreover, rescaling the random walk and passing to the limit, we get the most famous Paul Levy's arcsine law for Brownian motion.

Nevertheless, the result is not so simple. By modifying slightly the counting process of the positive terms of the random walk, as done by Chung & Feller in [1], an alternative sojourn time for the walk $(S_m)_{m\in\mathbb{N}}$ in $\mathbb{Z}^\dagger$ up to time n can be defined:

$$\tilde{T}_n = \sum_{m=1}^{n} \delta_m \text{ with } \delta_m = \begin{cases} 1 & \text{if } (S_m > 0) \text{ or } (S_m = 0 \text{ and } S_{m-1} > 0), \\ 0 & \text{if } (S_m < 0) \text{ or } (S_m = 0 \text{ and } S_{m-1} < 0). \end{cases} \tag{1.1}$$

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In $\tilde{T}_n$, one counts each time m such that $S_m > 0$ and only those times such that $S_m = 0$ which correspond to a downstep: $S_{m-1} = 1$. This convention is described in [1] and, as written therein– *"The elegance of the results to be announced depends on this convention"* (*sic*)–, it produces a remarkable result. Indeed, in this case, the probability distribution of this sojourn time takes the following simple form: for even integers m, n such that $0 \le m \le n$,

$$\mathbb{P}\{\tilde{T}_n = m\} = \frac{1}{2^n}\binom{m}{m/2}\binom{(n-m)}{(n-m)/2}.$$

In addition, also the conditioned random variable $(\tilde{T}_n|S_n = 0)$, for even n, is very simple: it is the uniform distribution on $\{0, 2, \dots, n-2, n\}$. The random variables $(\tilde{T}_n|S_n > 0)$ and $(\tilde{T}_n|S_n < 0)$ admit remarkable distributions as well; see, e.g., [14]. Hence, it is interesting to work with the joint distribution of $(\tilde{T}_n, S_n)$.

We observe that this approach could be adapted to a wider range of Markov chains.

1.2 Main results

In this paper, we consider a large class of Markov chains on a finite or denumerable state space $\mathcal{E}$ and we introduce the time T_n spent by such a chain in a fixed subset $E^\dagger$ of $\mathcal{E}$ up to a fixed time n. Inspired by the modified counting process (1.1), we also introduce an alternative sojourn time $\tilde{T}_n$. Let us recall that sojourn times in a fixed state play a fundamental role in the framework of general Markov chains and potential theory.

We develop a methodology for computing the probability distributions of T_n and $\tilde{T}_n$ via generating functions (Theorems 3.1 and 3.2). The technique is the following: by applying the Markov property, we write out linear systems for the generating functions of T_n and $\tilde{T}_n$. Though it seems difficult to solve these systems explicitly, nevertheless they could be numerically solved. We refer the reader to the famous book [5, Chap. XVI], to [10], [11], [17], [22] for an overview on Markov chains and to the recent book [6] for generating functions.

Next, we apply the general results we obtained to the case of (L, R)-random walk (Theorems 4.1, 4.2 and 4.3). We exhibit explicit results in the particular cases where $L = R = 1$, namely that of nearest random walk with possible stagnation (Theorems 5.1 and 5.2), and where $L = R = 2$, the case of the two-nearest random walk (Theorem 6.1). In this latter, we illustrate the matrix approach which can be completely carried out.

In the case of the usual Bernoulli random walk on $\mathbb{Z}$, the determination of the generating function of the sojourn time in $\mathbb{Z}^\dagger$ goes through that of the first hitting time of 0. In this particular example, level 0 acts as a boundary for $\mathbb{Z}^\dagger$ in the ordered set $\mathbb{Z}$. This observation leads us naturally to define in our context of Markov chains on $\mathcal{E}$ a kind of boundary, E° say, for the set of interest $E^\dagger$ which is appropriate to our problem (see Section 2). This is the reason why we restrict our study to Markov chains satisfying Assumptions (A_1) and (A_2) (see Section 2). Let us mention that, as in the case of Bernoulli walk, entrance times play an important role in the analysis.

1.3 Background and motivation

Our motivations come–among others–from biology. Certain stochastic models of genomic sequences are based on random walks with discrete bounded jumps. For instance, DNA and protein sequences, which are made of nucleotides or amino acids, can be modelled by Markov chains whose state space is a finite alphabet. Biological sequence analysis can be performed thanks to a powerful indicator: the so-called local score which is a functional of some random walk with discrete bounded jumps. This indicator plays an important role for studying alignements of two sequences, or for detecting particular functional properties of a sequence by assessing the statistical significance of an observed local score; see, e.g., [15], [16] and references therein.

Another example in biology concerns the study of micro-domains on a plasmic membrane in a cellular medium. The plasmic membrane is a place of interactions between the cell and its direct external environment. A naive stochastic model consists in viewing several kinds of constituents (the so-called ligands and receptors) as random walks evolving on the membrane. The time that ligands and receptors bind during a fixed amount of time plays an important role as a measurement of affinity/sentivity of ligands for receptors. It corresponds to the sojourn time in a suitable set for a certain random walk; see, e.g., [13].

Let us mention other fields of applications where entrance times, exit times, sojourn times for various random walks are decisive variables: finance, insurance, econometrics, reliability, management, queues, telecommunications, epidemiology, population dynamics...

More theoretically, several authors considered (L, R)-random walks (i.e., with integer-valued jumps lying between $-L$ and R where L and R are two positive integers) in the context of random walks in random environment. When fixing the environment, the quenched law they deal with is associated with a (L, R)-random walk. Let us quote for instance a pioneer work [12]; next [3] and [4] where the jumps are $+2$ and -1 (viz. $(1, 2)$-random walk); [2] and recent papers [7], [8], [9], corresponding to the particular cases $L = 1$ or $R = 1$, namely the jumps are 1 in one direction and greater than 1 in the other direction. Many other references can be found therein.

In all the aforementioned papers, one of the main motivation is the study of recurrence/transience and the statement of a law of large numbers for random walk in random environment. Important tools for tackling this study are passage time, exit time from a bounded interval, ladder times, excursions... in the quenched (i.e. fixed) environment. For instance, in [8], the authors study passage time and sojourn time above a level before reaching another one. In [7], the authors are able to compute generating function of the exit time of $(2,2)$-random walk (see also [2] for the case of $(1,2)$-random walk). In particular, in [9], we can read: *"These exit probabilities [in deterministic environment] play an important role in the offspring distribution of the branching structure, which can be expressed in terms of the environment" (sic).*

1.4 Plan of the paper

The paper is organized as follows. In Section 2 we introduce the settings. In particular, inspired by (1.1), we elaborate an alternative counting process. In Section 3 we consider several generating functions and, in particular, in Theorems 3.1 and 3.2, we describe a method for computing the generating functions of (T_n, X_n) and $(\tilde{T}_n, X_n)$ for a general Markov chain satisfying Assumptions (A_1) and (A_2). Since the proofs of Theorem 3.1 and 3.2 are quite technical, we postpone them to Section 7 as well as those of auxiliary results. In Section 4 we apply the methodology developed for general Markov chains to the class of (L,R)-random walks by adopting a matrix approach. Finally Section 5 and Section 6 are devoted to the more striking examples of ordinary random walks and symmetric $(2,2)$-random walks.

2 Settings

Let $(X_m)_{m\in\mathbb{N}}$ be an homogeneous Markov chain on a state space $\mathcal{E}$ (which is assumed to be finite or denumerable) and let $E^\dagger$, E^{o} be subsets of $\mathcal{E}$ with $E^{\mathrm{o}} \subset E^\dagger$. We set $E^+ = E^\dagger \backslash E^{\mathrm{o}}$, $E^- = \mathcal{E}\backslash E^\dagger$ and $E^\pm = \mathcal{E}\backslash E^{\mathrm{o}} = E^+ \cup E^-$. Actually, we partition the state space into $\mathcal{E} = E^+ \cup E^{\mathrm{o}} \cup E^-$. We will use the classical convention that $\min(\varnothing) = +\infty$ and that $\sum_{\ell=p}^{q} = 0$ if $p > q$. Throughout the paper, the letters i, j, k will denote generic space variables in $\mathcal{E}$ and ℓ, m, n will denote generic time variables in $\mathbb{N}$. We also introduce the classical conditional probabilities $\mathbb{P}_i\{\cdots\} = \mathbb{P}\{\cdots|X_0 = i\}$ and we set $p_{ij} = \mathbb{P}_i\{X_1 = j\}$ for any states $i, j \in \mathcal{E}$.

2.1 Entrance times

It will be convenient to introduce the first entrance times $\tau^{\mathrm{o}}, \tau^\dagger, \tau^+, \tau^-, \tau^\pm$ in $E^{\mathrm{o}}, E^\dagger, E^+, E^-, E^\pm$ respectively:

$$
\begin{aligned}
\tau^{\mathrm{o}} &= \min\{m \in \mathbb{N}^* : X_m \in E^{\mathrm{o}}\},\\
\tau^\dagger &= \min\{m \in \mathbb{N}^* : X_m \in E^\dagger\},\\
\tau^+ &= \min\{m \in \mathbb{N}^* : X_m \in E^+\},\\
\tau^- &= \min\{m \in \mathbb{N}^* : X_m \in E^-\},\\
\tau^\pm &= \min\{m \in \mathbb{N}^* : X_m \in E^\pm\}.
\end{aligned}
$$

We plainly have $\tau^\dagger \le \tau^{\mathrm{o}}$ and $\tau^\pm = \tau^+ \wedge \tau^-$. Additionally, we will use the first entrance time in $E^\pm$ after time τ^{o}:

$$\tilde{\tau}^\pm = \min\{m \ge \tau^{\mathrm{o}} : X_m \in E^\pm\}.$$

As a mnemonic, our settings write in the example of the classical Bernoulli random walk on $\mathcal{E} = \mathbb{Z}$, with the choices $E^{\mathrm{o}} = \{0\}$, $E^\dagger = \mathbb{Z}^\dagger = \{0, 1, 2, \dots\}$, $E^+ = \mathbb{Z}^+ = \{1, 2, \dots\}$, $E^- = \mathbb{Z}^- = \{\dots, -2, -1\}$ and $E^\pm = \mathbb{Z}\backslash\{0\}$, as

$$
\begin{aligned}
\tau^{\mathrm{o}} &= \min\{m \in \mathbb{N}^* : X_m = 0\},\\
\tau^\dagger &= \min\{m \in \mathbb{N}^* : X_m \ge 0\},\\
\tau^+ &= \min\{m \in \mathbb{N}^* : X_m > 0\},\\
\tau^- &= \min\{m \in \mathbb{N}^* : X_m < 0\},\\
\tau^\pm &= \min\{m \in \mathbb{N}^* : X_m \ne 0\},\\
\tilde{\tau}^\pm &= \min\{m \ge \tau^{\mathrm{o}} : X_m \ne 0\}.
\end{aligned}
$$

We make the following assumptions on the sets $E^\dagger$ and E^{o}:

(A_1) if $X_0 \in E^-$, then $\tau^{\mathrm{o}} = \tau^\dagger$. This means that the chain starting out of $E^\dagger$ enters $E^\dagger$ necessarily by passing through E^{o};

(A_2) if $X_0 \in E^+$, then $\tau^{\mathrm{o}} \le \tau^- - 1$ or, equivalently, $\tau^- \ge \tau^{\mathrm{o}} + 1$. This means that the chain starting in E^+ exits $E^\dagger$ necessarily by passing through E^{o}.

Roughly speaking, E^{o} acts as a kind of 'boundary' of $E^\dagger$, while E^+ acts as a kind of 'interior' of $E^\dagger$. These assumptions are motivated by the example of integer-valued (L, R)-random walks for which jumps are bounded from above by R and from below by $-L$ ($L, R \in \mathbb{N}^*$, L for *left*, R for *right*, the jumps lie in $\{-L, -L+1, \dots, R-1, R\}$). If we consider the sojourn time above level 0, we are naturally dealing with a 'thick' boundary above 0: $E^{\mathrm{o}} = \{0, 1, \dots, M\}$ where M is the maximum of L and R. Section 4 is devoted to this class of random walks.

2.2 Sojourn times

We consider the sojourn time of $(X_m)_{m\in\mathbb{N}}$ in $E^\dagger$ up to a fixed time $n \in \mathbb{N}$: $T_0 = 0$ and, for $n \ge 1$,

$$T_n = \#\{m \in \{1, \dots n\} : X_m \in E^\dagger\} = \sum_{m=1}^{n} \mathbb{1}_{E^\dagger}(X_m).$$

The random variable T_n counts the indices $m \in \{1, \dots, n\}$ for which $X_m \in E^\dagger$. Of course, we have $0 \le T_n \le n$. Inspired by the observation mentioned within the introduction concerning the alternative counting process (1.1), we define another sojourn time consisting in counting the $m \in \{1, \dots, n\}$ such that

- either $X_m \in E^+$,
- or $X_m \in E^{\mathrm{o}}$ and $X_{m-1} \in E^+$,
- or $X_m, X_{m-1} \in E^{\mathrm{o}}$, $X_{m-2} \in E^+$,

$\vdots$

- or $X_m, \dots, X_2 \in E^{\mathrm{o}}$, $X_1 \in E^+$.

Roughly speaking, we count the X_m lying in E^+ or the X_m lying in E^{o} 'coming' from a previous point lying in E^+. Let us introduce the following events: $B_1 = A_{0,1} = \{X_1 \in E^+\}$ and for any $m \in \mathbb{N}\backslash\{0, 1\}$,

$$\begin{aligned} A_{0,m} &= \{X_m \in E^+\}, \\ A_{\ell,m} &= \{X_m, X_{m-1}, \dots, X_{m-\ell+1} \in E^{\mathrm{o}}\} \cap \{X_{m-\ell} \in E^+\} \quad \text{for } \ell \in \{1, \dots, m-1\}, \\ B_m &= A_{0,m} \cup A_{1,m} \cup \cdots \cup A_{m-1,m}. \end{aligned}$$

In other terms, we consider the sojourn time defined by $\tilde{T}_0 = 0$ and for $n \in \mathbb{N}^*$, setting $\delta_m = \mathbb{1}_{B_m}$,

$$\tilde{T}_n = \#\{m \in \{1, \dots n\} : \delta_m = 1\} = \sum_{m=1}^{n} \delta_m.$$

It is interesting to know when, conversely, we do not count X_m through this process. This boils down to characterize the complement of B_m. For this, we set also $B'_1 = A'_{0,1} = \{X_1 \in E^- \cup E^{\mathrm{o}}\}$ and for any $m \in \mathbb{N}\backslash\{0, 1\}$,

$$\begin{aligned} A'_{0,m} &= \{X_m \in E^-\}, \\ A'_{\ell,m} &= \{X_m, X_{m-1}, \dots, X_{m-\ell+1} \in E^{\mathrm{o}}\} \cap \{X_{m-\ell} \in E^-\} \quad \text{for } \ell \in \{1, \dots, m-2\}, \\ A'_{m-1,m} &= \{X_m, X_{m-1}, \dots, X_2 \in E^{\mathrm{o}}\} \cap \{X_1 \in E^- \cup E^{\mathrm{o}}\}, \\ B'_m &= A'_{0,m} \cup A'_{1,m} \cup \cdots \cup A'_{m-1,m}. \end{aligned}$$

We have the following property the proof of which is postponed to Section 7.

Proposition 2.1 *For any $m \in \mathbb{N}^*$, the set B'_m is the complementary of B_m: $B'_m = B^c_m$.*

Remark 2.1 *The sets $A_{\ell,m}$, $0 \le \ell \le m-1$, are disjoint two by two, and the same holds for $A'_{\ell,m}$, $0 \le \ell \le m-1$. Hence, from Proposition 2.1, we deduce the following identities:*

$$\delta_m = \sum_{\ell=0}^{m} \mathbb{1}_{A_{\ell,m}} = \mathbb{1}_{B_m} = 1 - \mathbb{1}_{B^c_m} = 1 - \mathbb{1}_{B'_m} = 1 - \sum_{\ell=0}^{m} \mathbb{1}_{A'_{\ell,m}}.$$

Our aim is to provide a methodology for deriving the joint probability distributions of (T_n, X_n) and $(\tilde{T}_n, X_n)$. For this, we develop a method for computing the generating functions $\sum_{\substack{m,n\in\mathbb{N}:\\ m\le n}} \mathbb{P}_i\{T_n = m, X_n \in F\}\, x^m y^{n-m}$ (Subsection 3.2) and $\sum_{\substack{m,n\in\mathbb{N}:\\ m\le n}} \mathbb{P}_i\{\tilde{T}_n = m, X_n \in F\}\, x^m y^{n-m}$ (Subsection 3.5) for any subset F of $\mathcal{E}$. When $F = \mathcal{E}$, these quantities are simply related to the probability distributions of T_n and $\tilde{T}_n$. When $F = \{i\}$ for a fixed state $i \in \mathcal{E}$, this yields the probability distributions of the sojourn times in $E^\dagger$ up to time n for the 'bridge' (i.e., the pinned Markov chain) $(X_m|X_n = i)_{m\in\{0,\dots,n\}}$.

Actually, concerning $\tilde{T}_n$, we will only focus on the situation where $X_0 \in E^{\mathrm{o}}, X_1 \in E^{\pm}$ for lightening the paper and facilitating the reading. That is, we will only provide a way for computing the probabilities $\mathbb{P}_i\{X_1 \in E^{\pm}, \tilde{T}_n = m, X_n \in F\}$ for $i \in E^{\mathrm{o}}$ (Theorem 3.2). The study of $\tilde{T}_n$ subject to the complementary conditions $X_0, X_1 \in E^{\mathrm{o}}$ or $X_0 \notin E^{\mathrm{o}}$ could be carried out by using the results obtained under the previous conditions. We explain below with few details how to proceed.

- Under the condition $X_0, X_1 \in E^{\mathrm{o}}$, we consider the first exit time from E^{o}, namely $\tau^{\pm}$. Then, the first terms of the chain satisfy $X_0, X_1, \dots, X_{\tau^{\pm}-1} \in E^{\mathrm{o}}$ and $X_{\tau^{\pm}} \in E^{\pm}$ and we have $\tilde{T}_{\tau^{\pm}-1} = 0$. Viewing the chain from time $\tau^{\pm} - 1$, the facts $X_{\tau^{\pm}-1} \in E^{\mathrm{o}}, X_{\tau^{\pm}} \in E^{\pm}$ are the conditions analogous to $X_0 \in E^{\mathrm{o}}, X_1 \in E^{\pm}$.
- Under the condition $X_0 \notin E^{\mathrm{o}}$, we consider the first entrance time in E^{o}, and next the first exit time from E^{o}, namely $\tilde{\tau}^{\pm}$. Then, the first terms of the chain satisfy $X_0, X_1, \dots, X_{\tau^{\mathrm{o}}-1} \in E^{\pm}$, $X_{\tau^{\mathrm{o}}}, \dots, X_{\tilde{\tau}^{\pm}-1} \in E^{\mathrm{o}}$ and $X_{\tilde{\tau}^{\pm}} \in E^{\pm}$ and we have $\tilde{T}_{\tilde{\tau}^{\pm}-1} = \tilde{\tau}^{\pm} - 1$ if $X_0 \in E^{+}$ and $\tilde{T}_{\tilde{\tau}^{\pm}-1} = 0$ if $X_0 \in E^{-}$. From time $\tilde{\tau}^{\pm} - 1$, the facts $X_{\tilde{\tau}^{\pm}-1} \in E^{\mathrm{o}}, X_{\tilde{\tau}^{\pm}} \in E^{\pm}$ are the conditions analogous to $X_0 \in E^{\mathrm{o}}, X_1 \in E^{\pm}$.

In both situations, we could perform the computations by appealing to identities similar to (7.10) (which are used for proving Theorem 3.2) with the help of Markov property. We will let the reader write the details.

3 Generating functions

3.1 Generating functions of $X, \tau^{\mathrm{o}}, \tau^{\dagger}, \tau^{+}, \tau^{-}$

Let us introduce the following generating functions: for any states $i, j \in \mathcal{E}$, any subset $F \subset \mathcal{E}$ and any real number x such that the following series converge,

$$G_{ij}(x) = \sum_{m=0}^{\infty} \mathbb{P}_i\{X_m = j\}\, x^m, \quad G_i(x) = \sum_{m=0}^{\infty} \mathbb{P}_i\{X_m \in F\}\, x^m = \sum_{j\in F} G_{ij}(x),$$
$$H^{\mathrm{o}}_{ij}(x) = \sum_{m=1}^{\infty} \mathbb{P}_i\{\tau^{\mathrm{o}} = m, X_{\tau^{\mathrm{o}}} = j\}\, x^m = \mathbb{E}_i\big(x^{\tau^{\mathrm{o}}} \mathbb{1}_{\{X_{\tau^{\mathrm{o}}}=j\}}\big),$$
$$H^{\dagger}_{ij}(x) = \mathbb{E}_i\big(x^{\tau^{\dagger}} \mathbb{1}_{\{X_{\tau^{\dagger}}=j\}}\big), \quad H^{+}_{ij}(x) = \mathbb{E}_i\big(x^{\tau^{+}} \mathbb{1}_{\{X_{\tau^{+}}=j\}}\big), \quad H^{-}_{ij}(x) = \mathbb{E}_i\big(x^{\tau^{-}} \mathbb{1}_{\{X_{\tau^{-}}=j\}}\big).$$

In the above notations, the indices i and j refer to starting points and arrival points or entrance locations while superscripts $\mathrm{o}, \dagger, +, -$ refer to the entrance in the respective sets $E^{\mathrm{o}}, E^{\dagger}, E^{+}, E^{-}$. Moreover, for lightening the settings, when writing $X_{\tau^u} = j$ in $H^u_{ij}(x)$, $u \in \{\mathrm{o}, \dagger, +, -\}$, we implicitly restrict this event to the condition that $\tau^u < +\infty$ and we omit to write this condition explicitly.

For determining $G_{ij}(x)$, the standard method consists in using the well-known Chapman-Kolmogorov equation: for any $m \in \mathbb{N}$, $\ell \in \{0, \dots, m\}$ and $i, j \in \mathcal{E}$,

$$\mathbb{P}_i\{X_m = j\} = \sum_{k\in\mathcal{E}} \mathbb{P}_i\{X_\ell = k\}\, \mathbb{P}_k\{X_{m-\ell} = j\}. \tag{3.1}$$

In particular, by choosing $\ell = 1$ in (3.1), we have, for $m \ge 1$, that

$$\mathbb{P}_i\{X_m = j\} = \sum_{k\in\mathcal{E}} p_{ik}\, \mathbb{P}_k\{X_{m-1} = j\}$$

and we get that

$$G_{ij}(x) = \delta_{ij} + \sum_{m=1}^{\infty} \left(\sum_{k\in\mathcal{E}} p_{ik}\, \mathbb{P}_k\{X_{m-1} = j\} \right) x^m = \delta_{ij} + \sum_{k\in\mathcal{E}} p_{ik} \sum_{m=1}^{\infty} \mathbb{P}_k\{X_{m-1} = j\}\, x^m.$$

We obtain the famous backward Kolmogorov equation

$$G_{ij}(x) = \delta_{ij} + x \sum_{k\in\mathcal{E}} p_{ik} G_{kj}(x) \quad \text{for } i, j \in \mathcal{E}. \tag{3.2}$$

Similarly, by choosing $\ell = m-1$ with $m \geq 1$ in (3.1), we get the forward Kolmogorov equation

$$G_{ij}(x) = \delta_{ij} + x \sum_{k\in\mathcal{E}} p_{kj} G_{ik}(x) \quad \text{for } i,j \in \mathcal{E}. \tag{3.3}$$

For determining $H^{\mathrm{o}}_{ij}(x)$, we observe that for $i \in \mathcal{E}$, $j \in E^{\mathrm{o}}$ and $m \in \mathbb{N}^*$, if $X_0 = i$ and $X_m = j$, then the chain has entered E^{o} between times 1 and m. In symbols, appealing to the strong Markov property,

$$\mathbb{P}_i\{X_m = j\} = \mathbb{P}_i\{\tau^{\mathrm{o}} \leq m, X_m = j\} = \sum_{\ell=1}^{m} \sum_{k\in E^{\mathrm{o}}} \mathbb{P}_i\{\tau^{\mathrm{o}} = \ell, X_{\tau^{\mathrm{o}}} = k\}\, \mathbb{P}_k\{X_{m-\ell} = j\}.$$

Therefore,

$$\begin{aligned}
G_{ij}(x) &= \mathbb{P}_i\{X_0 = j\} + \sum_{m=1}^{\infty} \left(\sum_{\ell=1}^{m} \sum_{k\in E^{\mathrm{o}}} \mathbb{P}_i\{\tau^{\mathrm{o}} = \ell, X_{\tau^{\mathrm{o}}} = k\}\, \mathbb{P}_k\{X_{m-\ell} = j\} \right) x^m \\
&= \delta_{ij} + \sum_{\ell=1}^{\infty} \sum_{k\in E^{\mathrm{o}}} \mathbb{P}_i\{\tau^{\mathrm{o}} = \ell, X_{\tau^{\mathrm{o}}} = k\} \sum_{m=\ell}^{\infty} \mathbb{P}_k\{X_{m-\ell} = j\}\, x^m \\
&= \delta_{ij} + \sum_{\ell=1}^{\infty} \sum_{k\in E^{\mathrm{o}}} \mathbb{P}_i\{\tau^{\mathrm{o}} = \ell, X_{\tau^{\mathrm{o}}} = k\}\, x^{\ell} G_{kj}(x) \\
&= \delta_{ij} + \sum_{k\in E^{\mathrm{o}}} \mathbb{E}_i\big(x^{\tau^{\mathrm{o}}} \mathbb{1}_{\{X_{\tau^{\mathrm{o}}} = k\}}\big) G_{kj}(x).
\end{aligned}$$

We get the equation

$$G_{ij}(x) = \delta_{ij} + \sum_{k\in E^{\mathrm{o}}} H^{\mathrm{o}}_{ik}(x) G_{kj}(x) \quad \text{for } i \in \mathcal{E}, j \in E^{\mathrm{o}}. \tag{3.4}$$

Let us point out that, due to Assumptions (A_1) and (A_2), (3.4) holds true for $i \in E^+$, $j \in E^-$, and also for $i \in E^-$, $j \in E^\dagger$. In the same way, we have that

$$G_{ij}(x) = \delta_{ij} + \sum_{k\in E^\dagger} H^{\dagger}_{ik}(x) G_{kj}(x) \quad \text{for } i \in \mathcal{E}, j \in E^\dagger, \tag{3.5}$$

$$G_{ij}(x) = \delta_{ij} + \sum_{k\in E^+} H^{+}_{ik}(x) G_{kj}(x) \quad \text{for } i \in \mathcal{E}, j \in E^+, \tag{3.6}$$

$$G_{ij}(x) = \delta_{ij} + \sum_{k\in E^-} H^{-}_{ik}(x) G_{kj}(x) \quad \text{for } i \in \mathcal{E}, j \in E^-. \tag{3.7}$$

Moreover, if $i \in E^-$, then, by Assumption (A_1), $\tau^\dagger = \tau^{\mathrm{o}}$ which entails that $H^\dagger_{ij}(x) = H^{\mathrm{o}}_{ij}(x)$. If $i \in E^+$, then, by Assumption (A_2), $\tau^\dagger = 1$ which entails that $H^\dagger_{ij}(x) = p_{ij}\, x$. If $i \in E^{\mathrm{o}}$,

$$H^\dagger_{ij}(x) = \sum_{k\in E^\dagger} \mathbb{E}_i\big(x^{\tau^\dagger} \mathbb{1}_{\{X_1 = k, X_{\tau^\dagger} = j\}}\big) + \sum_{k\in E^-} \mathbb{E}_i\big(x^{\tau^\dagger} \mathbb{1}_{\{X_1 = k, X_{\tau^\dagger} = j\}}\big).$$

If $X_1 \in E^\dagger$, then $\tau^\dagger = 1$ while if $X_1 \in E^-$, then $\tau^\dagger = \tau^{\mathrm{o}}$. Now, let us introduce the first hitting time of $E^\dagger$ by $(X_m)_{m\in\mathbb{N}}$: $\tau^{\dagger\prime} = \min\{m \in \mathbb{N} : X_m \in E^\dagger\}$. Of course, we have $\tau^{\dagger\prime} = 0$ if $X_0 \in E^\dagger$, $\tau^{\dagger\prime} = \tau^\dagger$ if $X_0 \in E^-$ and $\tau^{\dagger\prime}$ is related to $\tau^\dagger$ according to $\tau^\dagger = 1 + \tau^{\dagger\prime} \circ \theta_1$ where θ_1 is the usual shift operator (acting as $X_m \circ \theta_1 = X_{m+1}$). Moreover, $X_{\tau^\dagger} = X_{\tau^{\dagger\prime}} \circ \theta_1$. With these settings at hands, thanks to the Markov property, we obtain $\mathbb{E}_i\big(x^{\tau^\dagger} \mathbb{1}_{\{X_1 = k, X_{\tau^\dagger} = j\}}\big) = p_{ik}\, x\, \mathbb{E}_k\big(x^{\tau^{\dagger\prime}} \mathbb{1}_{\{X_{\tau^{\dagger\prime}} = j\}}\big)$ which yields that

$$\mathbb{E}_i\big(x^{\tau^\dagger} \mathbb{1}_{\{X_1 = k, X_{\tau^\dagger} = j\}}\big) = \begin{cases} \delta_{jk}\, p_{ik}\, x & \text{if } k \in E^\dagger, \\ p_{ik}\, x\, \mathbb{E}_k\big(x^{\tau^{\mathrm{o}}} \mathbb{1}_{\{X_{\tau^{\mathrm{o}}} = j\}}\big) & \text{if } k \in E^-. \end{cases}$$

As a result, we see that the function $H^\dagger_{ij}$ can be expressed by means of H^{o}_{ij} according to

$$H^\dagger_{ij}(x) = \begin{cases} H^{\mathrm{o}}_{ij}(x) & \text{if } i \in E^-, \\ p_{ij}\, x & \text{if } i \in E^+, \\ x\left(p_{ij} + \displaystyle\sum_{k\in E^-} p_{ik} H^{\mathrm{o}}_{kj}(x) \right) & \text{if } i \in E^{\mathrm{o}}. \end{cases} \tag{3.8}$$

In the sequel of the paper, we will also use the generating functions below. Set, for any $i \in \mathcal{E}$ and $j \in E^{\mathrm{o}}$,

$$H_{ij}^{\mathrm{o}\dagger}(x) = \mathbb{E}_i\big(x^{\tau^{\mathrm{o}}} \mathbb{1}_{\{X_1 \in E^\dagger, X_{\tau^{\mathrm{o}}} = j\}}\big),$$
$$H_{ij}^{\mathrm{o}+}(x) = \mathbb{E}_i\big(x^{\tau^{\mathrm{o}}} \mathbb{1}_{\{X_1 \in E^+, X_{\tau^{\mathrm{o}}} = j\}}\big),$$
$$H_{ij}^{\mathrm{o}-}(x) = \mathbb{E}_i\big(x^{\tau^{\mathrm{o}}} \mathbb{1}_{\{X_1 \in E^-, X_{\tau^{\mathrm{o}}} = j\}}\big).$$

In short, $H_{ij}^{uv}(x)$, $u, v \in \{\mathrm{o}, \dagger, +, -\}$, is related to time τ^u and the first step $X_1 \in E^v$. We propose a method for calculating $H_{ij}^{\mathrm{o}\dagger}(x)$, $H_{ij}^{\mathrm{o}+}(x)$ and $H_{ij}^{\mathrm{o}-}(x)$. For this, we introduce the first hitting time of E^{o} by $(X_m)_{m\in\mathbb{N}}$: $\tau^{\mathrm{o}\prime} = \min\{m \in \mathbb{N} : X_m \in E^{\mathrm{o}}\}$. Of course, we have $\tau^{\mathrm{o}\prime} = 0$ if $X_0 \in E^{\mathrm{o}}$, $\tau^{\mathrm{o}\prime} = \tau^{\mathrm{o}}$ if $X_0 \notin E^{\mathrm{o}}$ and $\tau^{\mathrm{o}\prime}$ is related to τ^{o} according to $\tau^{\mathrm{o}} = 1 + \tau^{\mathrm{o}\prime} \circ \theta_1$. Moreover, $X_{\tau^{\mathrm{o}}} = X_{\tau^{\mathrm{o}\prime}} \circ \theta_1$. With these settings at hands, thanks to the Markov property, we obtain, for any $i \in \mathcal{E}$ and $j \in E^{\mathrm{o}}$, that

$$H_{ij}^{\mathrm{o}\dagger}(x) = x \sum_{k\in E^\dagger} p_{ik}\,\mathbb{E}_k\big(x^{\tau^{\mathrm{o}\prime}} \mathbb{1}_{\{X_{\tau^{\mathrm{o}\prime}} = j\}}\big) = x\left(\sum_{k\in E^{\mathrm{o}}} p_{ik}\delta_{kj} + \sum_{k\in E^+} p_{ik}\,\mathbb{E}_k\big(x^{\tau^{\mathrm{o}}} \mathbb{1}_{\{X_{\tau^{\mathrm{o}}} = j\}}\big)\right)$$

which simplifies into

$$H_{ij}^{\mathrm{o}\dagger}(x) = x\left(p_{ij} + \sum_{k\in E^+} p_{ik} H_{kj}^{\mathrm{o}}(x)\right). \tag{3.9}$$

Similarly,

$$H_{ij}^{\mathrm{o}+}(x) = x \sum_{k\in E^+} p_{ik} H_{kj}^{\mathrm{o}}(x), \quad H_{ij}^{\mathrm{o}-}(x) = x \sum_{k\in E^-} p_{ik} H_{kj}^{\mathrm{o}}(x). \tag{3.10}$$

3.2 Generating function of T_n

Now, we introduce the generating function of the numbers $\mathbb{P}_i\{T_n = m, X_n \in F\}$, $m, n \in \mathbb{N}$: set, for any $i \in \mathcal{E}$ and any real numbers x, y such that the following series converges,

$$K_i(x, y) = \sum_{\substack{m,n\in\mathbb{N}:\\ m\le n}} \mathbb{P}_i\{T_n = m, X_n \in F\}\, x^m y^{n-m}.$$

In the theorem below, we provide a way for computing $K_i(x, y)$.

Theorem 3.1 *The generating function K_i, $i \in \mathcal{E}$, satisfies the equation*

$$K_i(x, y) = K_i(x, 0) + K_i(0, y) + \sum_{j\in E^{\mathrm{o}}} \left(H_{ij}^{\mathrm{o}\dagger}(x) + \frac{x}{y} H_{ij}^{\mathrm{o}-}(y)\right) K_j(x, y) - \sum_{j\in E^{\mathrm{o}}} H_{ij}^{\mathrm{o}\dagger}(x) K_j(x, 0) - \mathbb{1}_F(i) \tag{3.11}$$

where, for any $i \in \mathcal{E}$,

$$K_i(x, 0) = G_i(x) - \sum_{j\in E^-} H_{ij}^-(x) G_j(x), \quad K_i(0, y) = G_i(y) - \sum_{j\in E^\dagger} H_{ij}^\dagger(y) G_j(y), \tag{3.12}$$

and where the functions H_{ij}^-, $j \in E^-$, and $H_{ij}^\dagger$, $j \in E^\dagger$, are given by (3.5) and (3.7), and the functions $H_{ij}^{\mathrm{o}\dagger}$ and $H_{ij}^{\mathrm{o}-}$, $j \in E^{\mathrm{o}}$ are given by (3.9) and (3.10).

Remark 3.1 *If we choose $F = \mathcal{E}$, for all state i in $\mathcal{E}$, we simply have $G_i(x) = \frac{1}{1-x}$ and (3.12) yields that*

$$K_i(x, 0) = \frac{1 - \mathbb{E}_i\big(x^{\tau^-}\big)}{1 - x}, \quad K_i(0, y) = \frac{1 - \mathbb{E}_i\big(y^{\tau^\dagger}\big)}{1 - y}.$$

Remark 3.2 *If E^{o} reduces to one point i_0, then, for $i = i_0$, (3.11) immediately yields*

$$K_{i_0}(x, y) = \frac{\big(1 - H_{i_0 i_0}^{\mathrm{o}\dagger}(x)\big) K_{i_0}(x, 0) + K_{i_0}(0, y) - \mathbb{1}_F(i_0)}{1 - H_{i_0 i_0}^{\mathrm{o}\dagger}(x) - \frac{x}{y} H_{i_0 i_0}^{\mathrm{o}-}(y)}.$$

Moreover, $H_{i_0 i_0}^{\mathrm{o}\dagger}(x)$ and $H_{i_0 i_0}^{\mathrm{o}-}(y)$ can be computed thanks to (3.9) and (3.10) with the aid of $H_{k i_0}^{\mathrm{o}}(x) = G_{k i_0}(x)/G_{i_0 i_0}(x)$ which comes from (3.4).

Let us mention that due to (3.11), it is enough to know $K_i(x, y)$ only for $i \in E^{\mathrm{o}}$ to derive $K_i(x, y)$ for $i \in \mathcal{E}\backslash E^{\mathrm{o}}$. Indeed, we have the connections below.

Proposition 3.1 *For $i \in \mathcal{E}\backslash E^{\mathrm{o}}$, $K_i(x,y)$ can be expressed by means of $K_j(x,y)$, $j \in E^{\mathrm{o}}$, according to the following identities: if $i \in E^+$,*

$$K_i(x,y) = G_i(x) - \sum_{j\in E^{\mathrm{o}}} H_{ij}^{\mathrm{o}}(x)G_j(x) + \sum_{j\in E^{\mathrm{o}}} H_{ij}^{\mathrm{o}}(x)K_j(x,y) \tag{3.13}$$

and if $i \in E^-$,

$$K_i(x,y) = G_i(y) - \sum_{j\in E^{\mathrm{o}}} H_{ij}^{\mathrm{o}}(y)G_j(y) + \frac{x}{y}\sum_{j\in E^{\mathrm{o}}} H_{ij}^{\mathrm{o}}(y)K_j(x,y). \tag{3.14}$$

3.3 Generating functions of $\tau^\pm$ and $\tilde{\tau}^\pm$

From now on we restrict ourselves to the case where $X_0 \in E^{\mathrm{o}}$ and $X_1 \in E^\pm$. In Subsection 2.2, we indicate how to treat the complementary case. Let us introduce the following generating functions: for any $i,j \in E^{\mathrm{o}}$,

$$\begin{aligned}
H_{ij}^{\pm}(x) &= \mathbb{E}_i\big(x^{\tau^\pm-1}1\!\!1_{\{X_{\tau^\pm-1}=j\}}\big) = \sum_{m=1}^{\infty} \mathbb{P}_i\{\tau^\pm = m, X_{\tau^\pm-1}=j\}\,x^{m-1},\\
\tilde{H}_{ij}^{\pm+}(x) &= \mathbb{E}_i\big(x^{\tilde{\tau}^\pm-1}1\!\!1_{\{X_1\in E^+,X_{\tilde{\tau}^\pm-1}=j\}}\big) = \sum_{m=1}^{\infty} \mathbb{P}_i\{X_1\in E^+,\tilde{\tau}^\pm = m, X_{\tilde{\tau}^\pm-1}=j\}\,x^{m-1},\\
\tilde{H}_{ij}^{\pm-}(x) &= \mathbb{E}_i\big(x^{\tilde{\tau}^\pm-1}1\!\!1_{\{X_1\in E^-,X_{\tilde{\tau}^\pm-1}=j\}}\big) = \sum_{m=1}^{\infty} \mathbb{P}_i\{X_1\in E^-,\tilde{\tau}^\pm = m, X_{\tilde{\tau}^\pm-1}=j\}\,x^{m-1}.
\end{aligned}$$

By noticing that $\tilde{\tau}^\pm = \tau^{\mathrm{o}} + \tau^\pm\circ\theta_{\tau^{\mathrm{o}}}$ where $\theta_{\tau^{\mathrm{o}}}$ acts on $(X_m)_{m\in\mathbb{N}}$ as $X_m\circ\theta_{\tau^{\mathrm{o}}} = X_{m+\tau^{\mathrm{o}}}$ and by using the strong Markov property, we have that

$$\tilde{H}_{ij}^{\pm+}(x) = \sum_{k\in E^{\mathrm{o}}} \mathbb{E}_i\big(x^{\tau^{\mathrm{o}}}1\!\!1_{\{X_1\in E^+,X_{\tau^{\mathrm{o}}}=k\}}\big)\,\mathbb{E}_k\big(x^{\tau^\pm-1}1\!\!1_{\{X_{\tau^\pm-1}=j\}}\big),$$

and a similar expression holds for $\tilde{H}_{ij}^{\pm-}(x)$. In short, we have obtained that

$$\tilde{H}_{ij}^{\pm+}(x) = \sum_{k\in E^{\mathrm{o}}} H_{ik}^{\mathrm{o}+}(x)H_{kj}^{\pm}(x),\quad \tilde{H}_{ij}^{\pm-}(x) = \sum_{k\in E^{\mathrm{o}}} H_{ik}^{\mathrm{o}-}(x)H_{kj}^{\pm}(x),$$

and, by (3.10),

$$\tilde{H}_{ij}^{\pm+}(x) = x\sum_{k\in E^+,\ell\in E^{\mathrm{o}}} p_{ik}H_{k\ell}^{\mathrm{o}}(x)H_{\ell j}^{\pm}(x),\quad \tilde{H}_{ij}^{\pm-}(x) = x\sum_{k\in E^-,\ell\in E^{\mathrm{o}}} p_{ik}H_{k\ell}^{\mathrm{o}}(x)H_{\ell j}^{\pm}(x). \tag{3.15}$$

Hence, we need to evaluate $H_{ij}^{\pm}(x)$ for $i,j\in E^{\mathrm{o}}$. We note that, if $X_1\in E^\pm$, then $\tau^\pm = 1$ and $X_{\tau^\pm-1} = X_0$, so that we get

$$\begin{aligned}
H_{ij}^{\pm}(x) &= \mathbb{E}_i\big(x^{\tau^\pm-1}1\!\!1_{\{X_1\in E^\pm,X_{\tau^\pm-1}=j\}}\big) + \mathbb{E}_i\big(x^{\tau^\pm-1}1\!\!1_{\{X_1\in E^{\mathrm{o}},X_{\tau^\pm-1}=j\}}\big)\\
&= \delta_{ij}\mathbb{P}_i\{X_1\in E^\pm\} + x\sum_{k\in E^{\mathrm{o}}}\mathbb{P}_i\{X_1=k\}\,\mathbb{E}_k\big(x^{\tau^\pm-1}1\!\!1_{\{X_{\tau^\pm-1}=j\}}\big).
\end{aligned}$$

As a result, we obtain the following equation: for $i,j\in E^{\mathrm{o}}$,

$$H_{ij}^{\pm}(x) = \delta_{ij}\mathbb{P}_i\{X_1\in E^\pm\} + x\sum_{k\in E^{\mathrm{o}}} p_{ik}H_{kj}^{\pm}(x). \tag{3.16}$$

Remark 3.3 *If E^{o} reduces to one point i_0, then (3.16) immediately yields $H_{i_0i_0}^{\pm}(x) = (1-p_{i_0i_0})/(1-p_{i_0i_0}x)$ which in turn yields that*

$$\tilde{H}_{i_0i_0}^{\pm+}(x) = \frac{1-p_{i_0i_0}}{1-p_{i_0i_0}x}\,H_{i_0i_0}^{\mathrm{o}+}(x),\quad \tilde{H}_{i_0i_0}^{\pm-}(x) = \frac{1-p_{i_0i_0}}{1-p_{i_0i_0}x}\,H_{i_0i_0}^{\mathrm{o}-}(x).$$

If we additionally impose the (more restrictive) condition that the Markov chain does not stay at its current location in E^{o}, that is, $p_{ii}=0$ for any $i\in E^{\mathrm{o}}$, then we simply have $H_{i_0i_0}^{\pm}(x)=1$, $\tilde{H}_{i_0i_0}^{\pm+}(x) = H_{i_0i_0}^{\mathrm{o}+}(x)$ and $\tilde{H}_{i_0i_0}^{\pm-}(x) = H_{i_0i_0}^{\mathrm{o}-}(x)$.

3.4 Generating functions of X, τ^+, τ^- subjected to $X_1 \in E^\pm$

In what follows we need the generating functions below: for any $i \in E^{\mathrm{o}}$, we set

$$G_i^+(x) = \sum_{m=0}^{\infty} \mathbb{P}_i\{X_1 \in E^+, X_m \in F\}\, x^m,$$
$$G_i^-(x) = \sum_{m=0}^{\infty} \mathbb{P}_i\{X_1 \in E^-, X_m \in F\}\, x^m;$$

for $i \in E^{\mathrm{o}}, j \in E^-$,

$$H_{ij}^{-+}(x) = \sum_{m=1}^{\infty} \mathbb{P}_i\{X_1 \in E^+, \tau^- = m, X_{\tau^-} = j\}\, x^m = \mathbb{E}_i\big(x^{\tau^-} \mathbb{1}_{\{X_1 \in E^+, X_{\tau^-} = j\}}\big);$$

for $i \in E^{\mathrm{o}}, j \in E^+$,

$$H_{ij}^{+-}(x) = \sum_{m=1}^{\infty} \mathbb{P}_i\{X_1 \in E^-, \tau^+ = m, X_{\tau^+} = j\}\, x^m = \mathbb{E}_i\big(x^{\tau^+} \mathbb{1}_{\{X_1 \in E^-, X_{\tau^+} = j\}}\big).$$

By the Markov property, we clearly have that

$$G_i^+(x) = \mathbb{1}_F(i)\mathbb{P}_i\{X_1 \in E^+\} + x \sum_{j \in E^+} p_{ij} G_j(x), \quad G_i^-(x) = \mathbb{1}_F(i)\mathbb{P}_i\{X_1 \in E^-\} + x \sum_{j \in E^-} p_{ij} G_j(x) \tag{3.17}$$

and

$$H_{ij}^{-+}(x) = x \sum_{k \in E^+} p_{ik} H_{kj}^-(x), \quad H_{ij}^{+-}(x) = x \sum_{k \in E^-} p_{ik} H_{kj}^+(x). \tag{3.18}$$

3.5 Generating function of $\tilde{T}_n$

Now, we introduce the generating function of the numbers $\mathbb{P}_i\{X_1 \in E^\pm, \tilde{T}_n = m, X_n \in F\}$, $m, n \in \mathbb{N}$: for any $i \in E^{\mathrm{o}}$,

$$\tilde{K}_i(x, y) = \sum_{\substack{m,n \in \mathbb{N}:\\ m \le n}} \mathbb{P}_i\{X_1 \in E^\pm, \tilde{T}_n = m, X_n \in F\}\, x^m y^{n-m}.$$

In the following theorem, we provide a way for computing $\tilde{K}_i(x, y)$.

Theorem 3.2 *The generating function $\tilde{K}_i$, $i \in E^{\mathrm{o}}$, satisfies the equation*

$$\begin{aligned}\tilde{K}_i(x, y) = {} & \tilde{K}_i(x, 0) + \tilde{K}_i(0, y) + \sum_{j \in E^{\mathrm{o}}} \big(\tilde{H}_{ij}^{\pm+}(x) + \tilde{H}_{ij}^{\pm-}(y)\big)\tilde{K}_j(x, y) \\ & - \sum_{j \in E^{\mathrm{o}}} \tilde{H}_{ij}^{\pm+}(x)\tilde{K}_j(x, 0) - \sum_{j \in E^{\mathrm{o}}} \tilde{H}_{ij}^{\pm-}(y)\tilde{K}_j(0, y) - \mathbb{1}_F(i)\mathbb{P}_i\{X_1 \in E^\pm\},\end{aligned} \tag{3.19}$$

where, for any $i \in E^{\mathrm{o}}$,

$$\begin{aligned}\tilde{K}_i(x, 0) &= G_i^+(x) - \sum_{j \in E^-} H_{ij}^{-+}(x) G_j(x) + \mathbb{1}_F(i)\mathbb{P}_i\{X_1 \in E^-\}, \\ \tilde{K}_i(0, y) &= G_i^-(y) - \sum_{j \in E^+} H_{ij}^{+-}(y) G_j(y) + \mathbb{1}_F(i)\mathbb{P}_i\{X_1 \in E^+\},\end{aligned} \tag{3.20}$$

and where the functions $\tilde{H}_{ij}^{\pm+}$ and $\tilde{H}_{ij}^{\pm-}$, $i, j \in E^{\mathrm{o}}$, are given by (3.15) and the functions H_{ij}^{-+}, $i \in E^{\mathrm{o}}$, $j \in E^-$, and H_{ij}^{+-}, $i \in E^{\mathrm{o}}$, $j \in E^+$, are given by (3.18).

Remark 3.4 *If we choose $F = \mathcal{E}$, for all state i in $\mathcal{E}$, we simply have $G_i^+(x) = \mathbb{P}_i\{X_1 \in E^+\}/(1 - x)$ and $G_i^-(x) = \mathbb{P}_i\{X_1 \in E^-\}/(1 - x)$, and (3.20) yields that*

$$\tilde{K}_i(x, 0) = \frac{1}{1 - x}\Big(\mathbb{P}_i\{X_1 \in E^\pm\} - x\,\mathbb{P}_i\{X_1 \in E^-\} - x \sum_{j \in E^+} p_{ij}\, \mathbb{E}_j\big(x^{\tau^-}\big)\Big),$$
$$\tilde{K}_i(0, y) = \frac{1}{1 - y}\Big(\mathbb{P}_i\{X_1 \in E^\pm\} - y\,\mathbb{P}_i\{X_1 \in E^+\} - y \sum_{j \in E^-} p_{ij}\, \mathbb{E}_j\big(y^{\tau^+}\big)\Big).$$

In view of Remark 3.3, if E^{o} reduces to one point i_0, then (3.19) immediately yields the explicit expression below.

Corollary 3.1 *If E^{o} reduces to one point i_0, then*

$$\tilde{K}_{i_0}(x,y) = \frac{\big(1-\tilde{H}^{\pm+}_{i_0i_0}(x)\big)\tilde{K}_{i_0}(x,0) + \big(1-\tilde{H}^{\pm-}_{i_0i_0}(y)\big)\tilde{K}_{i_0}(0,y) - \mathbb{1}_F(i_0)(1-p_{i_0i_0})}{1-\tilde{H}^{\pm+}_{i_0i_0}(x)-\tilde{H}^{\pm-}_{i_0i_0}(y)}.$$

Moreover, under the additional assumption that the Markov chain does not stay at its current location in E^{o}, that is $p_{ii}=0$ for any $i\in E^{\mathrm{o}}$, the generating function $\tilde{K}_{i_0}$ can be simplified into

$$\tilde{K}_{i_0}(x,y) = \frac{\big(1-H^{\mathrm{o}+}_{i_0i_0}(x)\big)\tilde{K}_{i_0}(x,0) + \big(1-H^{\mathrm{o}-}_{i_0i_0}(y)\big)\tilde{K}_{i_0}(0,y) - \mathbb{1}_F(i_0)}{1-H^{\mathrm{o}+}_{i_0i_0}(x)-H^{\mathrm{o}-}_{i_0i_0}(y)}.$$

4 Application to (L,R)-random walk

The algebraic equations (3.11) and (3.19) satisfied by the generating functions K_i and $\tilde{K}_i$, $i\in E^{\mathrm{o}}$, produce linear systems which may be rewritten in a matrix form possibly involving infinite matrices. In this section, we focus on the case of the (L,R)-random walk. In this case, the systems of interest consist of a finite number of equations and may be solved by using a matrix approach that we describe here. We guess that this approach provides a methodology which should be efficiently numerically implemented.

4.1 Settings

Let L,R be positive integers, $M=\max(L,R)$ and let $(U_\ell)_{\ell\in\mathbb{N}^*}$ be a sequence of independent identically distributed random variables with values in $\{-L,-L+1,\dots,R-1,R\}$. Put $\pi_i=\mathbb{P}\{U_1=i\}$ for $i\in\{-L,\dots,R\}$ and $\pi_i=0$ for $i\in\mathbb{Z}\backslash\{-L,\dots,R\}$. The common generating function of the U_ℓ's is given by

$$\mathbb{E}\big(y^{U_1}\big) = \sum_{j=-L}^{R}\pi_j\,y^j = y^{-L}\sum_{j=0}^{L+R}\pi_{j-L}\,y^j.$$

Let X_0 be an integer and set $X_m=X_0+\sum_{\ell=1}^m U_\ell$ for any $m\in\mathbb{N}^*$. Set $U_0=X_0$, and notice that X_m is the partial sum of the series $\sum U_\ell$. The Markov chain $(X_m)_{m\in\mathbb{N}}$ is a random walk defined on $\mathcal{E}=\mathbb{Z}$ with transition probabilities defined as $p_{ij}=\mathbb{P}\{X_{m+1}=j\,|\,X_m=i\}=\pi_{j-i}$. The jumps are bounded and we have that

$$p_{ij}=0 \text{ for } j\notin\{i-L,i-L+1,\dots,i+R\}. \tag{4.1}$$

We choose here

$$\begin{aligned}
E^{\mathrm{o}} &= \{0,1,\dots,M-1\},\\
E^{\dagger} &= \{0,1,\dots\},\\
E^{+} &= \{M,M+1,\dots\},\\
E^{-} &= \{\dots,-2,-1\}.
\end{aligned}$$

The settings of Section 2 can be rewritten in this context as

$$\begin{aligned}
T_n &= \#\{m\in\{1,\dots,n\}: X_m\geq 0\},\\
\tau^{\mathrm{o}} &= \min\{m\in\mathbb{N}^*: X_m\in\{0,1,\dots,M-1\}\},\\
\tau^{\dagger} &= \min\{m\in\mathbb{N}^*: X_m\geq 0\},\\
\tau^{+} &= \min\{m\in\mathbb{N}^*: X_m\geq M\},\\
\tau^{-} &= \min\{m\in\mathbb{N}^*: X_m\leq -1\}.
\end{aligned}$$

It is easy to see that Assumptions (A_1) and (A_2) are fulfilled. Indeed,

- if $X_0<0$, then $X_0,X_1,\dots,X_{\tau^\dagger-1}<0$ and $X_{\tau^\dagger}\geq 0$. Since $X_{\tau^\dagger}=X_{\tau^\dagger-1}+U_{\tau^\dagger}$, $X_{\tau^\dagger-1}\leq -1$ and $U_{\tau^\dagger}\leq R$, we have $X_{\tau^\dagger}\leq R-1\leq M-1$. Thus $\tau^\dagger=\tau^{\mathrm{o}}$;
- if $X_0\geq M$, then $X_0,X_1,\dots,X_{\tau^--1}\geq 0$ and $X_{\tau^-}<0$. Since $X_{\tau^--1}=X_{\tau^-}-U_{\tau^-}$, $X_{\tau^-}\leq -1$ and $U_{\tau^-}\geq -L$, we have $X_{\tau^--1}\leq L-1\leq M-1$. Thus $\tau^{\mathrm{o}}\leq\tau^--1$.

Now, we can observe the following connections between the foregoing times. Invariance by translation implies that upshooting level M (respectively downshooting level -1) when starting at a level $i\leq M-1$ (respectively

$i \geq 0$) is equivalent to upshooting level 0 (respectively downshooting level $M-1$) when starting at a level $i-M$ (respectively $i+M$). In symbols, we have that

$$\mathbb{P}_i\{\tau^- = m, X_{\tau^-} = j\} = \mathbb{P}_{i+M}\{\tau^{\mathrm{o}} = m, X_{\tau^{\mathrm{o}}} = j+M\} \quad \text{if } i \geq 0,$$
$$\mathbb{P}_i\{\tau^+ = m, X_{\tau^+} = j\} = \mathbb{P}_{i-M}\{\tau^{\mathrm{o}} = m, X_{\tau^{\mathrm{o}}} = j-M\} \quad \text{if } i \leq M-1,$$

and we deduce the identities below:

$$H^+_{ij}(x) = H^{\mathrm{o}}_{i-M\, j-M}(x) \text{ for any } i \leq M-1, \quad H^-_{ij}(x) = H^{\mathrm{o}}_{i+M\, j+M}(x) \text{ for any } i \geq 0. \tag{4.2}$$

Additionally, by (3.8),

$$H^\dagger_{ij}(x) = \begin{cases} H^{\mathrm{o}}_{ij}(x) & \text{if } i \leq -1, \\ p_{ij}\, x & \text{if } i \geq M, \\ x\left(p_{ij} + \displaystyle\sum_{k=-M}^{-1} p_{ik} H^{\mathrm{o}}_{kj}(x)\right) & \text{if } 0 \leq i \leq M-1. \end{cases} \tag{4.3}$$

4.2 Generating function of X

Recall that $G_{ij}(x) = \sum_{m=0}^{\infty} \mathbb{P}_i\{X_m = j\}\, x^m$. In the framework of random walk, we have the identity $G_{ij} = G_{0\,j-i}$ for any integers i, j; then it is convenient to introduce the notation $\Gamma_j = G_{0j}$ so that $G_{ij} = \Gamma_{j-i}$. We have the result below.

Proposition 4.1 *The generating function Γ_{j-i} admits the following representation: for $x \in (-1, 1)$,*

$$\Gamma_{j-i}(x) = \begin{cases} \displaystyle\sum_{\ell \in \mathcal{L}^-} \frac{z_\ell(x)^{i-j+L-1}}{P'_x(z_\ell(x))} & \text{if } i > j, \\ -\displaystyle\sum_{\ell \in \mathcal{L}^+} \frac{z_\ell(x)^{i-j+L-1}}{P'_x(z_\ell(x))} & \text{if } i \leq j, \end{cases} \tag{4.4}$$

where $z_\ell(x)$, $\ell \in \{1, \ldots, L+R\}$, are the roots of the polynomial $P_x : z \mapsto z^L - x \sum_{j=0}^{L+R} \pi_{j-L} z^j$ and

$$\mathcal{L}^+ = \{\ell \in \{1, \ldots, L+R\} : |z_\ell(x)| > 1\}, \quad \mathcal{L}^- = \{\ell \in \{1, \ldots, L+R\} : |z_\ell(x)| < 1\}.$$

PROOF

We introduce the double generating function of the numbers $\mathbb{P}_0\{X_m = j\}$, $m \in \mathbb{N}, j \in \mathbb{Z}$:

$$G(x, y) = \sum_{j \in \mathbb{Z}} \Gamma_j(x)\, y^j = \sum_{m=0}^{\infty} \left(\sum_{j \in \mathbb{Z}} \mathbb{P}_0\{X_m = j\}\, y^j\right) x^m = \sum_{m=0}^{\infty} \mathbb{E}_0\big(y^{X_m}\big)\, x^m = \sum_{m=0}^{\infty} \big[x\, \mathbb{E}\big(y^{U_1}\big)\big]^m$$

which can be simplified, for real numbers x, y such that $\big|x\, \mathbb{E}\big(y^{U_1}\big)\big| < 1$, into

$$G(x, y) = \frac{1}{1 - x\, \mathbb{E}(y^{U_1})} = \frac{y^L}{y^L - x \sum_{j=0}^{L+R} \pi_{j-L} y^j}.$$

Let us expand the rational fraction $y \mapsto G(x, y)$ into partial fractions. For this, we introduce the polynomial $P_x(z) = z^L - x \sum_{j=0}^{L+R} \pi_{j-L} z^j$ and assume that $|x| < 1$. We claim that the roots of P_x have a modulus different from 1. Else, if $\mathrm{e}^{\mathrm{i}\theta}$ was a root of P_x, we would have $\mathbb{E}\big(\mathrm{e}^{\mathrm{i}\theta U_1}\big) = 1/x$. This equality would simply entail that $|x| = 1/\big|\mathbb{E}\big(\mathrm{e}^{\mathrm{i}\theta U_1}\big)\big| \geq 1$ which contradicts our assumption on x. Denote by $z_\ell(x)$, $\ell \in \{1, \ldots, L+R\}$, the roots of P_x. By the foregoing discussion, we can separate the roots having modulus greater that 1 from those having modulus less that 1; they define the two sets $\mathcal{L}^+$ and $\mathcal{L}^-$. With these settings at hand, we can write out the expansion of $G(x, y)$ as

$$G(x, y) = \sum_{\ell=1}^{L+R} \frac{z_\ell(x)^L}{P'_x(z_\ell(x))} \frac{1}{y - z_\ell(x)}.$$

Next, by applying Taylor and Laurent series, we get

$$\frac{1}{y - z_\ell(x)} = \begin{cases} -\displaystyle\sum_{j=0}^{+\infty} \frac{y^j}{z_\ell(x)^{j+1}} & \text{if } \ell \in \mathcal{L}^+ \text{ and } |y| < \min_{\ell \in \mathcal{L}^+} |z_\ell(x)|, \\ \displaystyle\sum_{j=-\infty}^{-1} \frac{y^j}{z_\ell(x)^{j+1}} & \text{if } \ell \in \mathcal{L}^- \text{ and } |y| > \max_{\ell \in \mathcal{L}^-} |z_\ell(x)|, \end{cases}$$

and, since $G_{ij}(x) = \Gamma_{j-i}(x)$, we extract by identification (4.4). $\square$

4.3 Generating function of T_n and $\tilde{T}_n$

Our aim is to apply Theorems 3.1 and 3.2 to (L,R)-random walk. First, we make the following observations which will be useful:

- if $X_0 \geq 0$, then $X_{\tau^-} \in \{-M,\dots,-1\}$ and if $X_0 \leq M-1$, then $X_{\tau^\dagger} \in \{0,\dots,2M-1\}$ and $X_{\tau^+} \in \{M,\dots,2M-1\}$;
- if $\tau^\dagger > 1$, then $X_{\tau^\dagger} \leq M-1$, or, equivalently, if $X_{\tau^\dagger} \geq M$, then $\tau^\dagger = 1$. Thus $H^\dagger_{ij}(y) = p_{ij}\, y$ if $j \geq M$.

Now, we rephrase Theorem 3.1 in the present context.

Theorem 4.1 *The generating functions K_i, $i \in \{0,\dots,M-1\}$, satisfy the system*

$$\begin{aligned} K_i(x,y) = {} & x\sum_{j=0}^{M-1}\left(p_{ij} + \sum_{k=M}^{2M-1} p_{ik}H^{\circ}_{kj}(x) + \sum_{k=-M}^{-1} p_{ik}H^{\circ}_{kj}(y)\right)K_j(x,y) + K_i(x,0) + K_i(0,y) - \mathbb{1}_F(i) \\ & - x\sum_{j=0}^{M-1}\left(p_{ij} + \sum_{k=M}^{2M-1} p_{ik}H^{\circ}_{kj}(x)\right)K_j(x,0), \quad 0 \leq i \leq M-1, \end{aligned} \tag{4.5}$$

where

$$\begin{aligned} K_i(x,0) &= G_i(x) - \sum_{j=0}^{M-1} H^{\circ}_{i+M\,j}(x)G_{j-M}(x), \\ K_i(0,y) &= G_i(y) - y\sum_{j=0}^{2M-1} p_{ij}G_j(y) - y\sum_{j=0}^{M-1}\sum_{k=-M}^{-1} p_{ik}H^{\circ}_{kj}(y)G_j(y), \end{aligned} \tag{4.6}$$

and where the functions H°_{ij} solve the systems

$$\begin{aligned} &\sum_{k=0}^{M-1} H^{\circ}_{ik}(x)G_{kj}(x) = G_{ij}(x), \quad M \leq i \leq 2M-1,\ 0 \leq j \leq M-1, \\ &\sum_{k=0}^{M-1} H^{\circ}_{ik}(y)G_{kj}(y) = G_{ij}(y), \quad -M \leq i \leq -1,\ 0 \leq j \leq M-1. \end{aligned} \tag{4.7}$$

PROOF
By the observations mentioned at the beginning of this subsubsection, Equations (3.12) take the form, for $0 \leq i \leq M-1$,

$$\begin{aligned} K_i(x,0) &= G_i(x) - \sum_{j=-M}^{-1} H^{-}_{ij}(x)G_j(x), \\ K_i(0,y) &= G_i(y) - \sum_{j=0}^{M-1} H^{\dagger}_{ij}(y)G_j(y) - y\sum_{j=M}^{2M-1} p_{ij}G_j(y), \end{aligned}$$

which can be rewritten, due to (4.2) and (4.3), as (4.6). These latter contain the terms $H^{\circ}_{ij}(x)$, $M \leq i \leq 2M-1$, $0 \leq j \leq M-1$, and $H^{\circ}_{ij}(y)$, $-M \leq i \leq -1$, $0 \leq j \leq M-1$. By (4.1) and (3.4) we see that they solve Systems (4.7).

Next, in order to apply (3.11) to compute $K_i(x,y)$, we have to evaluate the quantities $H^{\circ\dagger}_{ij}(x)$ and $H^{\circ-}_{ij}(y)$. Because of (4.1), in view of (3.9) and (3.10), we have that

$$\begin{aligned} H^{\circ\dagger}_{ij}(x) &= x\left(p_{ij} + \sum_{\substack{k\in\mathbb{Z}:\\ M\leq k\leq i+R}} p_{ik}H^{\circ}_{kj}(x)\right) = x\left(p_{ij} + \sum_{k=M}^{2M-1} p_{ik}H^{\circ}_{kj}(x)\right), \quad 0 \leq i,j \leq M-1, \\ H^{\circ-}_{ij}(y) &= y\sum_{\substack{k\in\mathbb{Z}:\\ i-L\leq k\leq -1}} p_{ik}H^{\circ}_{kj}(y) = y\sum_{k=-M}^{-1} p_{ik}H^{\circ}_{kj}(y), \quad 0 \leq i,j \leq M-1. \end{aligned}$$

Finally, putting these last equalities into (3.11) yields (4.5). □

Now, let us rephrase Theorem 3.2 in the present context. Set $\varpi_i = \mathbb{P}_i\{X_1 \leq -1 \text{ or } X_1 \geq M\}$.

Theorem 4.2 *The generating functions $\tilde{K}_i$, $i \in \{0,\dots,M-1\}$, satisfy the system*

$$\begin{aligned}
\tilde{K}_i(x,y) = {} & \sum_{j=0}^{M-1}\left(x\sum_{k=M}^{2M-1}\sum_{\ell=0}^{M-1} p_{ik}H^{\circ}_{k\ell}(x)H^{\pm}_{\ell j}(x) + y\sum_{k=-M}^{-1}\sum_{\ell=0}^{M-1} p_{ik}H^{\circ}_{k\ell}(y)H^{\pm}_{\ell j}(y)\right)\tilde{K}_j(x,y) \\
& - x\sum_{j=0}^{M-1}\sum_{k=M}^{2M-1}\sum_{\ell=0}^{M-1} p_{ik}H^{\circ}_{k\ell}(x)H^{\pm}_{\ell j}(x)\tilde{K}_j(x,0) - y\sum_{j=0}^{M-1}\sum_{k=-M}^{-1}\sum_{\ell=0}^{M-1} p_{ik}H^{\circ}_{k\ell}(y)H^{\pm}_{\ell j}(y)\tilde{K}_j(0,y), \\
& + \tilde{K}_i(x,0) + \tilde{K}_i(0,y) - \mathbb{1}_F(i)\varpi_i, \quad 0 \le i \le M-1,
\end{aligned} \tag{4.8}$$

where

$$\begin{aligned}
\tilde{K}_i(x,0) &= x\sum_{j=M}^{2M-1} p_{ij}G_j(x) - x\sum_{j=0}^{M-1}\sum_{k=2M}^{3M-1} p_{i\,k-M}H^{\circ}_{kj}(x)G_{j-M}(x) + \mathbb{1}_F(i)\varpi_i, \\
\tilde{K}_i(0,y) &= y\sum_{j=-M}^{-1} p_{ij}G_j(y) - y\sum_{j=0}^{M-1}\sum_{k=-2M}^{-M-1} p_{i\,k+M}H^{\circ}_{kj}(y)G_{j+M}(y) + \mathbb{1}_F(i)\varpi_i,
\end{aligned} \tag{4.9}$$

and where the functions H°_{ij} and $H^{\pm}_{ij}$ solve the systems

$$\sum_{k=0}^{M-1} H^{\circ}_{ik}(x)G_{kj}(x) = G_{ij}(x), \quad M \le i \le 2M-1 \textit{ (resp. } 2M \le i \le 3M-1\textit{)},\ 0 \le j \le M-1, \tag{4.10}$$

$$\sum_{k=0}^{M-1} H^{\circ}_{ik}(y)G_{kj}(y) = G_{ij}(y), \quad -M \le i \le -1 \textit{ (resp. } -2M \le i \le -M-1\textit{)},\ 0 \le j \le M-1, \tag{4.11}$$

$$\sum_{k=0}^{M-1} (\delta_{ik} - p_{ik}x)H^{\pm}_{kj}(x) = \delta_{ij}\varpi_i, \quad 0 \le i,j \le M-1. \tag{4.12}$$

Proof

By the observations mentioned at the beginning of this subsubsection and by applying formula (3.17), Equations (3.20) take the form, for $0 \le i \le M-1$,

$$\begin{aligned}
\tilde{K}_i(x,0) &= G_i^{+}(x) - \sum_{j=-M}^{-1} H^{-+}_{ij}(x)G_j(x) + \mathbb{1}_F(i)\mathbb{P}_i\{X_1 \le -1\}, \\
\tilde{K}_i(0,y) &= G_i^{-}(y) - \sum_{j=M}^{2M-1} H^{+-}_{ij}(y)G_j(y) + \mathbb{1}_F(i)\mathbb{P}_i\{X_1 \ge M\}.
\end{aligned} \tag{4.13}$$

And, by (4.1), Equations (3.18) yield that

$$\begin{aligned}
H^{-+}_{ij}(x) &= x\sum_{k=M}^{2M-1} p_{ik}H^{-}_{kj}(x), \quad 0 \le i \le M-1,\ -M \le j \le -1, \\
H^{+-}_{ij}(y) &= y\sum_{k=-M}^{-1} p_{ik}H^{+}_{kj}(y), \quad 0 \le i \le M-1,\ M \le j \le 2M-1.
\end{aligned} \tag{4.14}$$

Actually, the terms $H^{-}_{ij}(x)$, $M \le i \le 2M-1$, $-M \le j \le -1$ and $H^{+}_{ij}(y)$, $-M \le i \le -1$, $M \le j \le 2M-1$ are directly related to the function H°_{ij} according to (4.2). Hence, putting (4.2) into (4.14), and next the obtained equality into Equations (4.13), these latter can be rewritten as (4.9).

On the other hand, by (3.4), the terms $H^{\circ}_{ij}(x)$, $2M \le i \le 3M-1$, $0 \le j \le M-1$, and $H^{\circ}_{ij}(y)$, $-2M \le i \le -M-1$, $0 \le j \le M-1$ lying in (4.9) solve Systems (4.10) and (4.11).

Additionally, by rewriting Equations (3.15) as

$$\tilde{H}^{\pm +}_{ij}(x) = x\sum_{k=M}^{2M-1}\sum_{\ell=0}^{M-1} p_{ik}H^{\circ}_{k\ell}(x)H^{\pm}_{\ell j}(x), \quad \tilde{H}^{\pm -}_{ij}(y) = y\sum_{k=-M}^{-1}\sum_{\ell=0}^{M-1} p_{ik}H^{\circ}_{k\ell}(y)H^{\pm}_{\ell j}(y),$$

and putting them into (3.19), we derive (4.8).

Finally, thanks to (3.16), we see that the terms $H^{\pm}_{kj}$ solve System (4.12). □

Since all the systems displayed in Theorem 4.1 and 4.2 are linear and include a finite number of equations, it is natural to solve them by matrix calculus. For this, we introduce the matrices below. We adopt the following intuitive settings: the subscripts $+,\ddagger$ (double-plus), $-,=$ (double-minus) refer to the appearance of the quantities $+M$, $+2M$, $-M$, $-2M$ in the generic term of the corresponding matrices, the subscript $\dagger$ refers to the set of indices $\{0,\dots,2M-1\}$ while the superscripts $+,-,\pm$ refer to the events $\{X_1\geq M\}$, $\{X_1\leq -1\}$, $\{X_1\leq -1 \text{ or } X_1\geq M\}$. So, with these conventions at hands, we set

$$\mathbf{I}=\big(\delta_{ij}\big)_{0\leq i,j\leq M-1},\quad \mathbf{I}^{\pm}=\big(\delta_{ij}\varpi_i\big)_{0\leq i,j\leq M-1},\quad \mathbf{1_F}=\big(\mathbb{1}_F(i)\big)_{0\leq i\leq M-1},\quad \mathbf{1}^{\pm}_{\mathbf{F}}=\big(\mathbb{1}_F(i)\varpi_i\big)_{0\leq i\leq M-1},$$

$$\mathbf{P}=\big(\pi_{j-i}\big)_{0\leq i,j\leq M-1},\quad \mathbf{\Gamma}(x)=\big(\Gamma_{j-i}(x)\big)_{0\leq i,j\leq M-1},$$

$$\mathbf{P}_-=\big(\pi_{j-i-M}\big)_{0\leq i,j\leq M-1},\quad \mathbf{P}_+=\big(\pi_{j-i+M}\big)_{0\leq i,j\leq M-1},\quad \mathbf{P}_\dagger=\big(\pi_{j-i}\big)_{0\leq i\leq M-1,0\leq j\leq 2M-1},$$

$$\mathbf{\Gamma}_-(x)=\big(\Gamma_{j-i-M}(x)\big)_{0\leq i,j\leq M-1},\quad \mathbf{\Gamma}_=(x)=\big(\Gamma_{j-i-2M}(x)\big)_{0\leq i,j\leq M-1},$$

$$\mathbf{\Gamma}_+(x)=\big(\Gamma_{j-i+M}(x)\big)_{0\leq i,j\leq M-1},\quad \mathbf{\Gamma}_\ddagger(x)=\big(\Gamma_{j-i+2M}(x)\big)_{0\leq i,j\leq M-1},$$

$$\mathbf{G}(x)=\big(G_i(x)\big)_{0\leq i\leq M-1},\quad \mathbf{G}_-(x)=\big(G_{i-M}(x)\big)_{0\leq i\leq M-1},$$

$$\mathbf{G}_+(x)=\big(G_{i+M}(x)\big)_{0\leq i\leq M-1},\quad \mathbf{G}_\dagger(x)=\big(G_i(x)\big)_{0\leq i\leq 2M-1},$$

$$\mathbf{K}(x,y)=\big(K_i(x,y)\big)_{0\leq i\leq M-1},\quad \tilde{\mathbf{K}}(x,y)=\big(\tilde{K}_i(x,y)\big)_{0\leq i\leq M-1}.$$

Example 4.1 *Consider the case where $M=2$, that is the case of the –at most– four nearest neighbours random walk (including the $(1,2)$-, $(2,1)$- and $(2,2)$-random walks). In this situation, the walk is characterized by the non negative numbers $\pi_{-2},\pi_{-1},\pi_0,\pi_1,\pi_2$ such that $\pi_{-2}+\pi_{-1}+\pi_0+\pi_1+\pi_2=1$ and we have $\pi_i=0$ for $i\in\mathbb{Z}\backslash\{-2,-1,0,1,2\}$. Recall the notation $\Gamma_j=G_{0j}$ for any integer j. Below, we rewrite the previous matrices:*

$$\mathbf{I}=\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad \mathbf{I}^{\pm}=\begin{pmatrix}1-\pi_0-\pi_1&0\\0&1-\pi_0-\pi_{-1}\end{pmatrix},\quad \mathbf{1_F}=\begin{pmatrix}\mathbb{1}_F(0)\\\mathbb{1}_F(1)\end{pmatrix},\quad \mathbf{1}^{\pm}_{\mathbf{F}}=\begin{pmatrix}\mathbb{1}_F(0)(1-\pi_0-\pi_1)\\\mathbb{1}_F(1)(1-\pi_0-\pi_{-1})\end{pmatrix},$$

$$\mathbf{P}=\begin{pmatrix}\pi_0&\pi_1\\\pi_{-1}&\pi_0\end{pmatrix},\quad \mathbf{P}_+=\begin{pmatrix}\pi_2&0\\\pi_1&\pi_2\end{pmatrix},\quad \mathbf{P}_-=\begin{pmatrix}\pi_{-2}&\pi_{-1}\\0&\pi_{-2}\end{pmatrix},\quad \mathbf{P}_\dagger=\begin{pmatrix}\pi_0&\pi_1&\pi_2&0\\\pi_{-1}&\pi_0&\pi_1&\pi_2\end{pmatrix},$$

$$\mathbf{\Gamma}(x)=\begin{pmatrix}\Gamma_0(x)&\Gamma_1(x)\\\Gamma_{-1}(x)&\Gamma_0(x)\end{pmatrix},\quad \mathbf{\Gamma}_+(x)=\begin{pmatrix}\Gamma_2(x)&\Gamma_3(x)\\\Gamma_1(x)&\Gamma_2(x)\end{pmatrix},\quad \mathbf{\Gamma}_-(x)=\begin{pmatrix}\Gamma_{-2}(x)&\Gamma_{-1}(x)\\\Gamma_{-3}(x)&\Gamma_{-2}(x)\end{pmatrix},$$

$$\mathbf{\Gamma}_=(x)=\begin{pmatrix}\Gamma_{-4}(x)&\Gamma_{-3}(x)\\\Gamma_{-5}(x)&\Gamma_{-4}(x)\end{pmatrix},\quad \mathbf{\Gamma}_\ddagger(x)=\begin{pmatrix}\Gamma_4(x)&\Gamma_5(x)\\\Gamma_3(x)&\Gamma_4(x)\end{pmatrix},$$

$$\mathbf{G}(x)=\begin{pmatrix}G_0(x)\\G_1(x)\end{pmatrix},\quad \mathbf{G}_+(x)=\begin{pmatrix}G_2(x)\\G_3(x)\end{pmatrix},\quad \mathbf{G}_-(x)=\begin{pmatrix}G_{-2}(x)\\G_{-1}(x)\end{pmatrix},\quad \mathbf{G}_\dagger(x)=\begin{pmatrix}G_0(x)\\G_1(x)\\G_2(x)\\G_3(x)\end{pmatrix},$$

$$\mathbf{K}(x,y)=\begin{pmatrix}K_0(x,y)\\K_1(x,y)\end{pmatrix},\quad \tilde{\mathbf{K}}(x,y)=\begin{pmatrix}\tilde{K}_0(x,y)\\\tilde{K}_1(x,y)\end{pmatrix}.$$

Theorem 4.3 *The generating matrix $\mathbf{K}$ admits the representation $\mathbf{K}(x,y)=\mathbf{D}(x,y)^{-1}\mathbf{N}(x,y)$ where*

$$\begin{aligned}\mathbf{D}(x,y)&=\mathbf{I}-x\big(\mathbf{P}+\mathbf{P}_+\mathbf{\Gamma}_-(x)\mathbf{\Gamma}(x)^{-1}+\mathbf{P}_-\mathbf{\Gamma}_+(y)\mathbf{\Gamma}(y)^{-1}\big),\\ \mathbf{N}(x,y)&=\ \big(\mathbf{I}-x\big(\mathbf{P}+\mathbf{P}_+\mathbf{\Gamma}_-(x)\mathbf{\Gamma}(x)^{-1}\big)\big)\big(\mathbf{G}(x)-\mathbf{\Gamma}_-(x)\mathbf{\Gamma}(x)^{-1}\mathbf{G}_-(x)\big)\\&\qquad+\mathbf{G}(y)-y\,\mathbf{P}_\dagger\mathbf{G}_\dagger(y)-y\,\mathbf{P}_-\mathbf{\Gamma}_+(y)\mathbf{\Gamma}(y)^{-1}\mathbf{G}(y)-\mathbf{1_F}.\end{aligned}\tag{4.15}$$

The generating matrix $\tilde{\mathbf{K}}$ admits the representation $\tilde{\mathbf{K}}(x,y)=\tilde{\mathbf{D}}(x,y)^{-1}\tilde{\mathbf{N}}(x,y)$ where

$$\begin{aligned}\tilde{\mathbf{D}}(x,y)&=\mathbf{I}-x\,\mathbf{P}_+\mathbf{\Gamma}_-(x)\mathbf{\Gamma}(x)^{-1}(\mathbf{I}-x\,\mathbf{P})^{-1}\mathbf{I}^{\pm}-y\,\mathbf{P}_-\mathbf{\Gamma}_+(y)\mathbf{\Gamma}(y)^{-1}(\mathbf{I}-y\,\mathbf{P})^{-1}\mathbf{I}^{\pm},\\ \tilde{\mathbf{N}}(x,y)&=\ \big(\mathbf{I}-x\,\mathbf{P}_+\mathbf{\Gamma}_-(x)\mathbf{\Gamma}(x)^{-1}(\mathbf{I}-x\,\mathbf{P})^{-1}\mathbf{I}^{\pm}\big)\times\Big(\mathbf{1}^{\pm}_{\mathbf{F}}+x\,\mathbf{P}_+\big(\mathbf{G}_+(x)-\mathbf{\Gamma}_=(x)\mathbf{\Gamma}(x)^{-1}\mathbf{G}_-(x)\big)\Big)\\&\quad+\big(\mathbf{I}-y\,\mathbf{P}_-\mathbf{\Gamma}_+(y)\mathbf{\Gamma}(y)^{-1}(\mathbf{I}-y\,\mathbf{P})^{-1}\mathbf{I}^{\pm}\big)\times\Big(\mathbf{1}^{\pm}_{\mathbf{F}}+y\,\mathbf{P}_-\big(\mathbf{G}_-(y)-\mathbf{\Gamma}_\ddagger(y)\mathbf{\Gamma}(y)^{-1}\mathbf{G}_+(y)\big)\Big)-\mathbf{1}^{\pm}_{\mathbf{F}}.\end{aligned}\tag{4.16}$$

In Section 5, we will apply this theorem to the ordinary random walk (corresponding to the case $M=1$), and in Subsubsection 6.2, we will apply Theorem 4.3 to the case of the symmetric $(2,2)$-random walk.

5 Ordinary random walk

In this section, we consider ordinary random walks on $\mathcal{E} = \mathbb{Z}$, that is, those for which $L = R = 1$ (and then $M = 1$). They are characterized by the three probabilities π_{-1}, π_0, π_1 that we respectively relabel as q, r, p for simplifying the settings:

$$q = \mathbb{P}\{U_1 = -1\}, \quad r = \mathbb{P}\{U_1 = 0\}, \quad p = \mathbb{P}\{U_1 = 1\}.$$

We have of course $p + q + r = 1$ and

$$p_{ij} = \begin{cases} q & \text{if } j = i-1, \\ r & \text{if } j = i, \\ p & \text{if } j = i+1, \\ 0 & \text{if } j \notin \{i-1, i, i+1\}. \end{cases}$$

We choose $E^{\text{o}} = \{0\}$, $E^\dagger = \{0, 1, 2, \dots\}$, $E^+ = \{1, 2, \dots\}$, $E^- = \{\dots, -2, -1\}$. Without loss of generality, we focus on the case where the starting point is located at 0, i.e., in the current settings, we choose $i = 0$. Our aim is to apply Theorem 4.3 to this example.

The roots of the polynomial $P_x(z) = -(px\, z^2 - (1 - rx)z + qx)$ are expressed as

$$z(x) = \frac{1 - rx - \sqrt{\Delta(x)}}{2px}, \quad \zeta(x) = \frac{1 - rx + \sqrt{\Delta(x)}}{2px}$$

where $\Delta(x) = (1 - rx)^2 - 4pqx^2$. They are chosen such that $|z(x)| < 1 < |\zeta(x)|$ for $x \in (0, 1)$. We have $z(x)\zeta(x) = q/p$ and $P_x'(z) = -2px\, z + (1 - rx)$. In particular, $P_x'(z(x)) = -P_x'(\zeta(x)) = \sqrt{\Delta(x)}$.

Formula (4.4) yields, for $x \in (0, 1)$, that

$$\Gamma_{j-i}(x) = \begin{cases} \dfrac{z(x)^{i-j}}{\sqrt{\Delta(x)}} & \text{if } i \geq j, \\[2ex] \dfrac{\zeta(x)^{i-j}}{\sqrt{\Delta(x)}} & \text{if } i \leq j. \end{cases}$$

In particular,

$$\Gamma_0(x) = \frac{1}{\sqrt{\Delta(x)}}, \quad \Gamma_{-1}(x) = \frac{z(x)}{\sqrt{\Delta(x)}}, \quad \Gamma_1(x) = \frac{1}{\zeta(x)\sqrt{\Delta(x)}} = \frac{pz(x)}{q\sqrt{\Delta(x)}}.$$

$$\Gamma_{-2}(x) = \frac{z(x)^2}{\sqrt{\Delta(x)}}, \quad \Gamma_2(x) = \frac{1}{\zeta(x)^2\sqrt{\Delta(x)}} = \frac{p^2 z(x)^2}{q^2\sqrt{\Delta(x)}}.$$

Next, we rewrite the matrices of Subsubsection 4.3 with the convention that for matrices with one entry, we omit the parentheses and assimilate them to numbers:

$$\mathbf{I} = 1, \quad \mathbf{I}^{\pm} = \mathbf{1}_{\mathbf{F}}^{\pm} = \mathbb{1}_F(0)(1 - r), \quad \mathbf{1}_{\mathbf{F}} = \mathbb{1}_F(0), \quad \mathbf{P} = r, \quad \mathbf{P}_+ = p, \quad \mathbf{P}_- = q, \quad \mathbf{P}_\dagger = \begin{pmatrix} r & p \end{pmatrix},$$

$$\mathbf{\Gamma}(x) = \Gamma_0(x), \quad \mathbf{\Gamma}_-(x) = \Gamma_{-1}(x), \quad \mathbf{\Gamma}_+(x) = \Gamma_1(x), \quad \mathbf{\Gamma}_=(x) = \Gamma_{-2}(x), \quad \mathbf{\Gamma}_\ddagger(x) = \Gamma_2(x),$$

$$\mathbf{K}(x, y) = K_0(x, y), \quad \tilde{\mathbf{K}}(x, y) = \tilde{K}_0(x, y).$$

For simplifying the forthcoming results, we will only consider the cases where $F = \mathbb{Z}$ and $F = \{0\}$. In the case $F = \mathbb{Z}$, we have $G_i(x) = 1/(1 - x)$ for any integer i and

$$\mathbf{G}(x) = \mathbf{G}_+(x) = \mathbf{G}_-(x) = \frac{1}{1 - x}, \quad \mathbf{G}_\dagger(x) = \frac{1}{1 - x}\begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

while in the case $F = \{0\}$, we have $G_i(x) = \Gamma_{-i}(x)$ for any integer i and

$$\mathbf{G}(x) = \Gamma_0(x), \quad \mathbf{G}_-(x) = \Gamma_1(x), \quad \mathbf{G}_+(x) = \Gamma_{-1}(x), \quad \mathbf{G}_\dagger(x) = \begin{pmatrix} \Gamma_0(x) \\ \Gamma_{-1}(x) \end{pmatrix}.$$

Theorem 4.3 provides the result below.

Theorem 5.1 *Set $\Delta(u) = (1 - ru)^2 - 4pqu^2$. The generating function K_0 admits the following expression:*

- *if $F = \mathbb{Z}$,*

$$K_0(x, y) = y\, \frac{(p - q)(x - y) + (1 - y)\sqrt{\Delta(x)} + (1 - x)\sqrt{\Delta(y)}}{(1 - x)(1 - y)\big(y - x + y\sqrt{\Delta(x)} + x\sqrt{\Delta(y)}\,\big)}; \tag{5.1}$$

- *if* $F = \{0\}$,

$$K_0(x,y) = \frac{2y}{y - x + y\sqrt{\Delta(x)} + x\sqrt{\Delta(y)}}. \tag{5.2}$$

Proof

Observe that $\Gamma_{-1}(x)/\Gamma_0(x) = z(x)$ and $\Gamma_1(x)/\Gamma_0(x) = 1/\zeta(x) = pz(x)/q$.

- In the case where $F = \mathbb{Z}$, in view of (4.15), we see that $K_0(x,y)$ can be written as $N(x,y)/D(x,y)$ where

$$\begin{aligned}(1-x)(1-y)N(x,y) &= (1-y)\big(1 - rx - px\,z(x)\big)\big(1 - z(x)\big)\\ &\quad + (1-x)\big(1 - (p+r)y - py\,z(y)\big) - (1-x)(1-y),\\ D(x,y) &= 1 - rx - px\,z(x) - px\,z(y).\end{aligned}$$

  Using the identity $px\,z(x)^2 = (1-rx)z(x) - qx$, straightforward computations lead to

$$\begin{aligned}2(1-x)(1-y)N(x,y) &= 2\big(1 - (q+r)x - (p+r)y + rxy - px(1-y)z(x) - py(1-x)z(y)\big)\\ &= (p-q)(x-y) + (1-y)\sqrt{\Delta(x)} + (1-x)\sqrt{\Delta(y)},\\ 2y\,D(x,y) &= y - x + y\sqrt{\Delta(x)} + x\sqrt{\Delta(y)}.\end{aligned}$$

  We immediately deduce (5.1).

- In the case where $F = \{0\}$, by (4.15), $K_0(x,y)$ can be written as $N(x,y)/D(x,y)$ where $D(x,y)$ has exactly the same expression as previously and

$$N(x,y) = \frac{1}{\sqrt{\Delta(x)}}\big(1 - rx - px\,z(x)\big)\Big(1 - \frac{p}{q}\,z(x)^2\Big) + \frac{1}{\sqrt{\Delta(y)}}\big(1 - ry - 2py\,z(y)\big) - 1.$$

  Thanks to $px\,z(x)^2 = (1-rx)z(x) - qx$, elementary computations give

$$\begin{aligned}\big(1 - rx - px\,z(x)\big)\Big(1 - \frac{p}{q}\,z(x)^2\Big) &= \frac{1}{2qx}\big(1 - rx - pxz(x)\big)\big(2qx - (1-rx)z(x)\big) = \sqrt{\Delta(x)},\\ 1 - ry - 2py\,z(y) &= \sqrt{\Delta(y)},\end{aligned}$$

  so that $N(x,y) = 1$ from which we extract (5.2).

□

Proposition 5.1 *For $F = \mathbb{Z}$, the following identity holds true:*

$$K_0(x,y) = K_0(x,0)K_0(0,y). \tag{5.3}$$

Proof

Putting $x = 0$ or $y \to 0$ into (5.1) and observing that $y - x + y\sqrt{\Delta(x)} + x\sqrt{\Delta(y)} \sim_{y\to 0} y\big(1 - rx + \sqrt{\Delta(x)}\,\big)$ entail that

$$K_0(x,0) = \frac{1 - (2q+r)x + \sqrt{\Delta(x)}}{(1-x)\big(1 - rx + \sqrt{\Delta(x)}\,\big)} \quad \text{and} \quad K_0(0,y) = \frac{1 - (2p+r)y + \sqrt{\Delta(y)}}{2(1-y)}.$$

By elementary calculations, we see that

$$\big((2p+r)x - 1 + \sqrt{\Delta(x)}\,\big)\big(1 - rx + \sqrt{\Delta(x)}\,\big) = 2px\big(1 - (2q+r)x + \sqrt{\Delta(x)}\,\big)$$

and $K_0(x,0)$ can be simplified into

$$K_0(x,0) = \frac{(2p+r)x - 1 + \sqrt{\Delta(x)}}{2px(1-x)}.$$

Therefore, the product of the two terms $K_0(x,0)$ and $K_0(0,y)$ is given by

$$K_0(x,0)K_0(0,y) = \frac{\big((2p+r)x - 1 + \sqrt{\Delta(x)}\,\big)\big(1 - (2p+r)y + \sqrt{\Delta(y)}\,\big)}{4px(1-x)(1-y)}. \tag{5.4}$$

Straightforward but tedious computations that we do not report here show that

$$\big((2p+r)x - 1 + \sqrt{\Delta(x)}\,\big)\big(1 - (2p+r)y + \sqrt{\Delta(y)}\,\big)\big(y - x + y\sqrt{\Delta(x)} + x\sqrt{\Delta(y)}\,\big)$$

$$= 4pxy\big((p-q)(x-y) + (1-y)\sqrt{\Delta(x)} + (1-x)\sqrt{\Delta(y)}\,\big).$$

Finally, comparing (5.1) and (5.4) ensures the validity of (5.3). □

Decomposition (5.3) may be appropriate for inverting the generating function K_0 in order to provide a closed form for the probability distribution of T_n. We will not go further in this direction.

Concerning $\tilde{K}_0$, for $i = 0$, we introduce the following settings:

$$\begin{aligned}
A(x,y) = &(1-rx)(1-ry)\Big(2(1-r)(1-qx-py-rxy) \\
&- \frac{1}{px}(1-y)\big(r(1-rx)^2 + (1-r)^2\,px\big) - \frac{1}{qy}(1-x)\big(r(1-ry)^2 + (1-r)^2\,qy\big)\Big), \\
B(x,y) = &\frac{1}{px}\,(1-y)(1-ry)\big(r(1-rx)^2 + (1-r)^2\,px\big), \\
C(x,y) = &\frac{1}{qy}\,(1-x)(1-rx)\big(r(1-ry)^2 + (1-r)^2\,qy\big).
\end{aligned}$$

The quantity $A(x,y)$ admits the following expansion:

$$A(x,y) = (1-rx)(1-ry)\Big(a_{11}xy + a_{10}x + a_{01}y + a_{1-1}\frac{x}{y} + a_{-11}\frac{y}{x} - a_{-10}\frac{1}{x} - a_{0-1}\frac{1}{y} + a_{00}\Big),$$

where the coefficients a_{ij} (of x^iy^j) are expressed by means of p, q, r as follows: by substituting $p + q = 1 - r$,

$$\begin{aligned}
a_{11} &= \frac{2r^3}{p} + \frac{2r^3}{q} + 2r^2 - 2r = 2r(1-r)\left(\frac{r^2}{pq} - 2\right), \\
a_{10} &= -\frac{r^3}{p} - \frac{2r^2}{q} + (1-r)^2 - 2q(1-r) = -r^2\left(\frac{r}{p} + \frac{2}{q}\right) + (p-q)(1-r), \\
a_{01} &= -\frac{r^3}{q} - \frac{2r^2}{p} + (1-r)^2 - 2p(1-r) = -r^2\left(\frac{r}{q} + \frac{2}{p}\right) + (q-p)(1-r), \\
a_{00} &= \frac{2r^2}{p} + \frac{2r^2}{q} - 2r^2 + 2r = 2r(1-r)\left(\frac{r}{pq} + 1\right), \\
a_{-10} &= -\frac{r}{p}, \quad a_{0-1} = -\frac{r}{q}, \quad a_{1-1} = \frac{r}{q}, \quad a_{-11} = \frac{r}{p}.
\end{aligned}$$

We also introduce

$$\begin{aligned}
A'(x,y) &= (1-rx)(1-ry)\left(\big(1-2r-r^2\big) + r\,\frac{(1-rx)^2}{2pqx^2} + r\,\frac{(1-ry)^2}{2pqy^2}\right), \\
B'(x,y) &= -\frac{r(1-ry)}{pqx^2}\,\big(pq(1-r)x^2 + (1-rx)^2\big), \\
C'(x,y) &= -\frac{r(1-rx)}{pqy^2}\,\big(pq(1-r)y^2 + (1-ry)^2\big).
\end{aligned}$$

Theorem 4.3 provides the result below.

Theorem 5.2 *The generating function $\tilde{K}_0$ admits the following expression:*

- *if $F = \mathbb{Z}$,*

$$\tilde{K}_0(x,y) = \frac{1}{(1-x)(1-y)}\,\frac{A(x,y) + B(x,y)\sqrt{\Delta(x)} + C(x,y)\sqrt{\Delta(y)}}{2r(1-rx)(1-ry) + (1-r)\big((1-ry)\sqrt{\Delta(x)} + (1-rx)\sqrt{\Delta(y)}\,\big)}; \tag{5.5}$$

- *if $F = \{0\}$,*

$$\tilde{K}_0(x,y) = \frac{A'(x,y) + B'(x,y)\sqrt{\Delta(x)} + C'(x,y)\sqrt{\Delta(y)}}{2r(1-rx)(1-ry) + (1-r)\big((1-ry)\sqrt{\Delta(x)} + (1-rx)\sqrt{\Delta(y)}\,\big)}. \tag{5.6}$$

Proof

- In the case where $F=\mathbb{Z}$, in view of (4.16), we see that $\tilde{K}_0(x,y)$ can be written as $\tilde{N}(x,y)/\tilde{D}(x,y)$ where

$$\tilde{N}(x,y)=\Big(1-\frac{1-r}{1-rx}\,px\,z(x)\Big)\Big((1-r)+\frac{px}{1-x}\left(1-z(x)^2\right)\Big)$$
$$+\Big(1-\frac{1-r}{1-ry}\,py\,z(y)\Big)\Big((1-r)+\frac{qy}{1-y}\left(1-\frac{p^2}{q^2}\,z(y)^2\right)\Big)-(1-r),$$
$$\tilde{D}(x,y)=1-\frac{1-r}{1-rx}\,px\,z(x)-\frac{1-r}{1-ry}\,py\,z(y).$$

Using the identity $px\,z(x)^2=(1-rx)z(x)-qx$, we easily get that

$$(1-x)(1-y)(1-rx)(1-ry)\tilde{N}(x,y)$$
$$=(1-y)(1-ry)\big(1-rx-(1-r)\,px\,z(x)\big)\big((1-r)-(1-rx)z(x)\big)$$
$$+(1-x)(1-rx)\big(1-ry-(1-r)\,py\,z(y)\big)\Big((1-r)-\frac{p}{q}(1-ry)z(y)\Big)$$
$$-(1-r)(1-x)(1-y)(1-rx)(1-ry),$$
$$(1-rx)(1-ry)\tilde{D}(x,y)$$
$$=(1-rx)(1-ry)-(1-r)(1-ry)px\,z(x)-(1-r)(1-rx)py\,z(y).$$

Straightforward computations lead to

$$2(1-x)(1-y)(1-rx)(1-ry)\tilde{N}(x,y)=A(x,y)+B(x,y)\sqrt{\Delta(x)}+C(x,y)\sqrt{\Delta(y)},$$
$$2(1-rx)(1-ry)\tilde{D}(x,y)=2r(1-rx)(1-ry)+(1-r)\big((1-ry)\sqrt{\Delta(x)}+(1-rx)\sqrt{\Delta(y)}\,\big),$$

from which we deduce (5.5).

- In the case where $F=\{0\}$, by (4.16), $\tilde{K}_0(x,y)$ can be written as $\tilde{N}(x,y)/\tilde{D}(x,y)$ where $\tilde{D}(x,y)$ has exactly the same expression as previously and

$$\tilde{N}(x,y)=\Big(1-\frac{1-r}{1-rx}\,px\,z(x)\Big)\Big[(1-r)+\frac{px\,z(x)}{\sqrt{\Delta(x)}}\Big(1-\frac{p}{q}\,z(x)^2\Big)\Big]$$
$$+\Big(1-\frac{1-r}{1-ry}\,py\,z(y)\Big)\Big[(1-r)+\frac{py\,z(y)}{\sqrt{\Delta(y)}}\Big(1-\frac{p}{q}\,z(y)^2\Big)\Big]-(1-r)$$

By replacing $z(x)$ by its expression, elementary computations give

$$\Big(1-\frac{1-r}{1-rx}\,px\,z(x)\Big)\Big[(1-r)+\frac{px\,z(x)}{\sqrt{\Delta(x)}}\Big(1-\frac{p}{q}\,z(x)^2\Big)\Big]=\Big(1-\frac{3}{2}\,r-\frac{1}{2}r^2+r\frac{(1-rx)^2}{2pqx^2}\Big)$$
$$-\frac{r}{2}\Big(\frac{1-rx}{pqx^2}+\frac{1-r}{1-rx}\Big)\sqrt{\Delta(x)},$$

so that $N(x,y)=1$ from which we extract (5.6).

□

In the case where $r=0$ (and $p+q=1$), that is, when the random walk does not stay at its current location, the generating function $\tilde{K}_0$ can be simplified. We write its expression below.

Corollary 5.1 *In the case where $r=0$, we have $\Delta(x)=\sqrt{1-4pqx^2}$ and the generating function $\tilde{K}_0$ is given by*

- *if $F=\mathbb{Z}$,*

$$\tilde{K}_0(x,y)=\frac{(p-q)(x-y)+(1-y)\sqrt{\Delta(x)}+(1-x)\sqrt{\Delta(y)}}{(1-x)(1-y)\big(\sqrt{\Delta(x)}+\sqrt{\Delta(y)}\,\big)};$$

- *if $F=\{0\}$,*

$$\tilde{K}_0(x,y)=\frac{1}{\sqrt{\Delta(x)}+\sqrt{\Delta(y)}}.$$

We retrieve the expressions obtained in [14]; especially in the case where $F=\mathbb{Z}$, the corresponding expression is rewritten in the form

$$\tilde{K}_0(x,y)=\frac{1}{\sqrt{\Delta(x)}+\sqrt{\Delta(y)}}\left(\frac{p-q+\sqrt{\Delta(x)}}{1-x}+\frac{q-p+\sqrt{\Delta(y)}}{1-y}\right)$$

which can be inverted in order to provide a closed form for the probability distribution of $\tilde{T}_n$ (see [14]).

6 Symmetric random walk

In this part, we focus on the particular random walks satisfying $L = R = M$, with steps lying in $\{-M, -M+1, \dots, M-1, M\}$, such that $\pi_i = \pi_{-i}$ for all integer i. In this case, the random walk $(X_m)_{m\in\mathbb{N}}$ is symmetric. We provide a representation of the generating function of τ^{o} which can be inserted into the matrices introduced in Section 4.

6.1 Generating function of X

The polynomial P_x takes the form

$$P_x(z) = z^M \left[1 - \pi_0 x - x \sum_{j=1}^{M} \pi_j \left(z^j + \frac{1}{z^j}\right)\right].$$

To simplify the discussion, we will suppose throughout Section 6 that all the roots of P_x are distinct.

We immediately see that its roots are inverse two by two so that the sets $\mathcal{L}^+$ and $\mathcal{L}^-$ have the same cardinality M. It is clear that there are M roots of modulus less than one, while their M inverses have modulus greater than one. So, we relabel the roots as $z_\ell(x)$ and $z_{\ell+M}(x) = 1/z_\ell(x)$, $\ell \in \{1, \dots, M\}$ with the convention that $|z_\ell(x)| < 1$. This yields $\mathcal{L}^- = \{1, \dots, M\}$ and $\mathcal{L}^+ = \{M+1, \dots, 2M\}$.

Because of the equality $P_x(z) = z^{2M} P_x(1/z)$, we have $P_x'(z) = 2M z^{2M-1} P_x(1/z) - z^{2M-2} P_x'(1/z)$ and this implies that

$$\frac{z_\ell(x)^{M-1}}{P_x'(z_\ell(x))} = -\frac{(1/z_\ell(x))^{M-1}}{P_x'(1/z_\ell(x))}.$$

Therefore, expressions (4.4) of $\Gamma_{j-i}(x)$ can be simplified and we get the generating function of X which is displayed below.

Proposition 6.1 *The generating function of X is characterized by the numbers $G_{ij}(x) = \Gamma_{j-i}(x)$, $i, j \in \mathbb{Z}$, $x \in (0,1)$, with*

$$\Gamma_{j-i}(x) = \sum_{m=0}^{\infty} \mathbb{P}_i\{X_m = j\}\, x^m = \sum_{\ell=1}^{M} \frac{z_\ell(x)^{M-1}}{P_x'(z_\ell(x))}\, z_\ell(x)^{|i-j|}. \tag{6.1}$$

Below, we give three examples of such random walks.

Example 6.1 *Let us consider the random walk with jump probabilities given by*

$$\begin{cases} \pi_i = c\dbinom{2M}{i+M} & \text{for } i \in \{-M, -M+1, \dots, -1, 1, \dots, M-1, M\}, \\[2ex] \pi_0 = 1 - c\left[4^M - \dbinom{2M}{M}\right], & \end{cases}$$

where c is a positive constant such that $c \le 1/\left[4^M - \binom{2M}{M}\right]$. For $c = 1/4^M$, we have $\pi_i = \binom{2M}{i+M}/4^M$ for any $i \in \{-M, -M+1, \dots, M-1, M\}$ and for $c = 1/\left[4^M - \binom{2M}{M}\right]$, we have $\pi_0 = 0$, that is, the walker never stays at its current position.

In this case,

$$\mathbb{E}\big(y^{U_1}\big) = \frac{1}{y^M}\left(c(y+1)^{2M} + (1 - c\,4^M)\, y^M\right),$$

and

$$G(x,y) = \frac{y^M}{P_x(y)} \quad \text{where} \quad P_x(z) = \big(1 - (1 - c\,4^M)\, x\big) z^M - c\,x(z+1)^{2M}.$$

Suppose that $x \in (0,1)$. The roots of P_x are those of the M quadratic equations

$$(z+1)^2 - e^{i\frac{2\pi}{M}r} \sqrt{\frac{1 - (1 - c\,4^M)\,x}{c\,x}}\; z = 0, \quad 0 \le r \le M-1.$$

We have that

$$P_x'(z_\ell(x)) = M\big(1 - (1 - c\,4^M)\,x\big) z_\ell(x)^{M-1} - 2Mc\,x(z_\ell(x)+1)^{2M-1} = M\big(1 - (1 - c\,4^M)\,x\big)\frac{1 - z_\ell(x)}{1 + z_\ell(x)}\, z_\ell(x)^{M-1}.$$

Expression (6.1) takes the form

$$\Gamma_{j-i}(x) = \frac{1}{M\,(1 - (1 - c\,4^M)\,x)} \sum_{\ell=1}^{M} \frac{1 + z_\ell(x)}{1 - z_\ell(x)}\, z_\ell(x)^{|i-j|}.$$

Example 6.2 *Let us consider the random walk with jump probabilities given by*

$$\begin{cases} \pi_i = c\rho^{|i|}\binom{M}{|i|} & \text{for } i \in \{-M, -M+1, \dots, -1, 1, \dots, M-1, M\}, \\ \pi_0 = 1 - 2c\left((\rho+1)^M - 1\right), \end{cases}$$

where c and ρ are positive constants such that $c \leq 1/\big(2(\rho+1)^M - 1\big)$. For $c = 1/\big(2(\rho+1)^M - 1\big)$, we have $\pi_0 = 0$, that is, the walker never stays at its current position.

In this case,

$$\mathbb{E}\big(y^{U_1}\big) = c\left((\rho y + 1)^M + \left(\frac{\rho}{y} + 1\right)^M - 2(\rho+1)^M\right) + 1,$$

and

$$G(x,y) = \frac{y^M}{P_x(y)} \quad \text{where} \quad P_x(z) = \left[1 - \left(1 - 2c(\rho+1)^M\right)x\right] z^M - cx\left[(\rho z^2 + z)^M + (z+\rho)^M\right].$$

Example 6.3 *Let us consider the random walk with jump probabilities given by*

$$\begin{cases} \pi_i = c & \text{for } i \in \{-M, -M+1, \dots, -1, 1, \dots, M-1, M\}, \\ \pi_0 = 1 - 2Mc, \end{cases}$$

where c is a positive constant such that $c \leq 1/(2M)$. This is a random walk to the $2M$ nearest neighbours with identically distributed jumps and a possible stay at the current position. For $c = 1/(2M)$, we have $\pi_0 = 0$, that is, the walker never stays at its current position. For $c = 1/(2M+1)$, each step put the walker to the $2M$ nearest neighbours or let it at its current position with identical probability.

In this case,

$$\mathbb{E}\big(y^{U_1}\big) = \big(1 - (2M+1)c\big) + c\,\frac{1 - y^{2M+1}}{y^M(1-y)},$$

and

$$G(x,y) = \frac{y^M}{P_x(y)} \quad \text{where} \quad P_x(z) = \frac{1}{1-z}\left[\big(1 - (1-(2M+1)c)x\big)\big(z^M - z^{M+1}\big) - cx\big(1 - z^{2M+1}\big)\right].$$

6.2 Symmetric $(2,2)$-random walk

In Example 4.1, we have considered the case of non-symmetric $(2,2)$-random walk. Now we have a look on the symmetric $(2,2)$-random walk (corresponding to the case $M = 2$) which is characterized by the three non-negative numbers π_0, π_1, π_2 such that $\pi_0 + 2\pi_1 + 2\pi_2 = 1$:

$$\pi_0 = \mathbb{P}\{U_1 = 0\}, \quad \pi_1 = \mathbb{P}\{U_1 = 1\} = \mathbb{P}\{U_1 = -1\}, \quad \pi_2 = \mathbb{P}\{U_1 = 2\} = \mathbb{P}\{U_1 = -2\}.$$

We have that

$$\mathbb{E}\big(y^{U_1}\big) = \pi_2\left(y^2 + \frac{1}{y^2}\right) + \pi_1\left(y + \frac{1}{y}\right) + \pi_0,$$

and $G(x,y) = y^2/P_x(y)$ where

$$P_x(z) = z^2 - x\left(\pi_2 z^4 + \pi_1 z^3 + \pi_0 z^2 + \pi_1 z + \pi_2\right) = z^2\left[1 - x\left(\pi_2\left(z^2 + \frac{1}{z^2}\right) + \pi_1\left(z + \frac{1}{z}\right) + \pi_0\right)\right].$$

Set $\delta(x) = (\pi_1 + 4\pi_2)^2 + 4\pi_2(1/x - 1)$. The roots of the polynomial P_x can be explicitly calculated by introducing the intermediate unknown $\zeta = z + 1/z$ and solving two quadratic equations. For $x \in (0,1)$, the roots are real and distinct, those of absolute value less than 1 are given by

$$\begin{aligned} z_1(x) &= -\frac{1}{4\pi_2}\left(\pi_1 - \sqrt{\delta(x)} + \sqrt{2}\sqrt{\pi_1^2 + 4\pi_1\pi_2 - 2\pi_2 + \frac{2\pi_2}{x} - \pi_1\sqrt{\delta(x)}}\right), \\ z_2(x) &= -\frac{1}{4\pi_2}\left(\pi_1 + \sqrt{\delta(x)} + \sqrt{2}\sqrt{\pi_1^2 + 4\pi_1\pi_2 - 2\pi_2 + \frac{2\pi_2}{x} + \pi_1\sqrt{\delta(x)}}\right), \end{aligned} \tag{6.2}$$

and the other ones are $z_3(x) = 1/z_1(x)$ and $z_4(x) = 1/z_2(x)$. Moreover, rewriting P_x as $P_x(z) = z^2 Q_x(z + 1/z)$ with $Q_x(\zeta) = 1 - x\left(\pi_2\zeta^2 + \pi_1\zeta + \pi_0 - 2\pi_2 - 1/x\right)$, we get that if z is a root of P_x, then

$$P_x'(z) = \left(z^2 - 1\right) Q'\left(z + \frac{1}{z}\right) = x\left(1 - z^2\right)\left(2\pi_2\left(z + \frac{1}{z}\right) + \pi_1\right)$$

which can be simplified into

$$P_x'(z_1(x)) = x\left(1 - z_1(x)^2\right)\sqrt{\delta(x)}, \quad P_x'(z_2(x)) = -x\left(1 - z_2(x)^2\right)\sqrt{\delta(x)}.$$

Identity (6.1) yields the generating function of X below.

Proposition 6.2 *The generating function of X is characterized by the numbers $G_{ij}(x) = \Gamma_{j-i}(x)$, $i, j \in \mathbb{Z}$, $x \in (0, 1)$, with*

$$\Gamma_{j-i}(x) = \frac{1}{x\sqrt{\delta(x)}}\left(\frac{z_1(x)^{|j-i|+1}}{1 - z_1(x)^2} - \frac{z_2(x)^{|j-i|+1}}{1 - z_2(x)^2}\right)$$

where $z_1(x)$ and $z_2(x)$ are displayed in (6.2).

Finally, we provide a representation of the generating functions of sojourn times T_n and $\tilde{T}_n$ in the case where $F = \mathbb{Z}$. We apply Formulas (4.15) and (4.16) with $M = 2$ and $F = \mathbb{Z}$.

Theorem 6.1 *The generating matrices of T_n and $\tilde{T}_n$ admit the respective representations*

$$\mathbf{K}(x, y) = \mathbf{D}(x, y)^{-1}\mathbf{N}(x, y) \quad \text{and} \quad \tilde{\mathbf{K}}(x, y) = \tilde{\mathbf{D}}(x, y)^{-1}\tilde{\mathbf{N}}(x, y)$$

where $\mathbf{D}(x, y)^{-1}, \mathbf{N}(x, y)$ are given by (7.11)–(7.12) further and $\tilde{\mathbf{D}}(x, y)^{-1}, \tilde{\mathbf{N}}(x, y)$ by (7.13)–(7.14).

The explicit expressions of matrices $\mathbf{K}(x, y)$ and $\tilde{\mathbf{K}}(x, y)$ involving cumbersome computations, we postpone the proof of Theorem 6.1 to Subsection 7.5.

7 Proofs of Theorem 3.1, Theorem 3.2 and Theorem 6.1

In this section we give the proofs of Theorem 3.1, Theorem 3.2 and all the auxiliary results stated in Sections 2 and 3 as well as the proof of Theorem 6.1. Subsections 7.1 and 7.2 concern time T_n while Subsections 7.3 and 7.4 concern time $\tilde{T}_n$; Subsection 7.5 concerns the case of $(2, 2)$-symmetric random walk.

7.1 Proof of Theorem 3.1

We first observe that for $n \in \mathbb{N}^*$,

- $T_n = 0$ if and only if for all $m \in \{1, \dots, n\}$, $X_m \in E^-$, that is if and only if $\tau^\dagger > n$;
- $T_n = n$ if and only if for all $m \in \{1, \dots, n\}$, $X_m \in E^\dagger$, that is if and only if $\tau^- > n$;
- $1 \le T_n \le n - 1$ if and only if there exists distinct integers $\ell, \ell' \in \{1, \dots, n\}$ such that $X_\ell \in E^\dagger$ and $X_{\ell'} \in E^-$. This is equivalent to saying that $\tau^- \le n$ and $\tau^\dagger \le n$.

In the last case, we have the three possibilities below:

- if $X_0 \in E^-$, then $\tau^{\mathrm{o}} = \tau^\dagger \le n$ by Assumption (A_1);
- if $X_0 \in E^+$, then $\tau^{\mathrm{o}} \le \tau^- - 1 \le n - 1$ by Assumption (A_2);
- if $X_0 \in E^{\mathrm{o}}$, the following possibilities occur:
  - if $X_1 \in E^{\mathrm{o}}$, then $\tau^{\mathrm{o}} = 1$;
  - if $X_1 \in E^+$, then $\tau^{\mathrm{o}} \le \tau^- - 1$. In this case, τ^{o} is the return time to E^{o};
  - if $X_1 \in E^-$, then $\tau^{\mathrm{o}} = \tau^\dagger$. In this case, τ^{o} is the return time to E^{o}.

This discussion entails, for any $i \in \mathcal{E}$, that

$$\begin{aligned}\mathbb{P}_i\{T_n = 0, X_n \in F\} &= \mathbb{P}_i\{\tau^\dagger > n, X_n \in F\} = \mathbb{P}_i\{X_n \in F\} - \mathbb{P}_i\{\tau^\dagger \le n, X_n \in F\} \\ &= \mathbb{P}_i\{X_n \in F\} - \sum_{\ell=1}^{n}\mathbb{P}_i\{\tau^\dagger = \ell, X_n \in F\}.\end{aligned}$$

From this, we deduce the following:

$$
\begin{aligned}
K_i(0,y) &= \sum_{n=0}^{\infty} \mathbb{P}_i\{T_n = 0, X_n \in F\}\, y^n \\
&= \sum_{n=0}^{\infty} \mathbb{P}_i\{X_n \in F\}\, y^n - \sum_{n=1}^{\infty}\sum_{\ell=1}^{n} \mathbb{P}_i\{\tau^\dagger = \ell, X_n \in F\}\, y^n \\
&= \sum_{j\in F}\left(\sum_{n=0}^{\infty} \mathbb{P}_i\{X_n = j\}\, y^n\right) - \sum_{\ell=1}^{\infty}\sum_{n=\ell}^{\infty}\sum_{j\in E^\dagger} \mathbb{P}_i\{\tau^\dagger = \ell, X_{\tau^\dagger} = j\}\, \mathbb{P}_j\{X_{n-\ell} \in F\}\, y^n \\
&= \sum_{j\in F} G_{ij}(y) - \sum_{j\in E^\dagger}\left(\sum_{\ell=1}^{\infty} \mathbb{P}_i\{\tau^\dagger = \ell, X_{\tau^\dagger} = j\}\, y^\ell\right)\left(\sum_{n=0}^{\infty} \mathbb{P}_j\{X_n \in F\}\, y^n\right) \\
&= G_i(y) - \sum_{j\in E^\dagger} H^\dagger_{ij}(y) G_j(y).
\end{aligned}
$$

This proves the second equality in (3.12). In a very similar way,

$$
\mathbb{P}_i\{T_n = n, X_n \in F\} = \mathbb{P}_i\{\tau^- > n, X_n \in F\} = \mathbb{P}_i\{X_n \in F\} - \sum_{\ell=1}^{n} \mathbb{P}_i\{\tau^- = \ell, X_n \in F\}.
$$

This entails that

$$
K_i(x,0) = \sum_{n=0}^{\infty} \mathbb{P}_i\{T_n = n, X_n \in F\}\, x^n = G_i(x) - \sum_{j\in E^-} H^-_{ij}(x) G_j(x)
$$

which proves the first equality in (3.12). Next, we compute $K_i(x,y)$:

$$
\begin{aligned}
K_i(x,y) &= \sum_{n=0}^{\infty} \mathbb{P}_i\{T_n = n, X_n \in F\}\, x^n + \sum_{n=0}^{\infty} \mathbb{P}_i\{T_n = 0, X_n \in F\}\, y^n - \mathbb{P}_i\{X_0 \in F\} \\
&\quad + \sum_{\substack{m,n\in\mathbb{N}:\\ 1\le m\le n-1}} \mathbb{P}_i\{T_n = m, X_n \in F\}\, x^m y^{n-m} \\
&= K_i(x,0) + K_i(0,y) + \sum_{\substack{m,n\in\mathbb{N}:\\ 1\le m\le n-1}} \mathbb{P}_i\{\tau^{\mathrm{o}} \le n, T_n = m, X_n \in F\}\, x^m y^{n-m} - \mathbb{1}_F(i). \qquad (7.1)
\end{aligned}
$$

We observe that on the set $\{\tau^{\mathrm{o}} \le n\}$

$$
T_n = \sum_{m=1}^{n} \mathbb{1}_{E^\dagger}(X_m) = T_{\tau^{\mathrm{o}}} + \sum_{m=\tau^{\mathrm{o}}+1}^{n} \mathbb{1}_{E^\dagger}(X_m) = T_{\tau^{\mathrm{o}}} + \sum_{m=1}^{n-\tau^{\mathrm{o}}} \mathbb{1}_{E^\dagger}(X_{m+\tau^{\mathrm{o}}}) = T_{\tau^{\mathrm{o}}} + T^{(\tau^{\mathrm{o}})}_{n-\tau^{\mathrm{o}}}
$$

where $T^{(\tau^{\mathrm{o}})}_{n-\tau^{\mathrm{o}}}$ is the sojourn time in $E^\dagger$ up to time $n-\tau^{\mathrm{o}}$ for the shifted chain $(X_{m+\tau^{\mathrm{o}}})_{m\in\mathbb{N}}$. Moreover,

- if $X_1 \in E^\dagger$, we have $X_1,\dots,X_{\tau^{\mathrm{o}}} \in E^\dagger$ and then $T_{\tau^{\mathrm{o}}} = \tau^{\mathrm{o}}$ which yields $T^{(\tau^{\mathrm{o}})}_{n-\tau^{\mathrm{o}}} = T_n - \tau^{\mathrm{o}}$;
- if $X_1 \in E^-$, we have $X_1,\dots,X_{\tau^{\mathrm{o}}-1} \in E^-, X_{\tau^{\mathrm{o}}} \in E^{\mathrm{o}}$; in this case $T_{\tau^{\mathrm{o}}} = 1$ and then $T^{(\tau^{\mathrm{o}})}_{n-\tau^{\mathrm{o}}} = T_n - 1$.

Consequently, for $m \in \{1,\dots,n-1\}$ and $i \in \mathcal{E}$,

$$
\begin{aligned}
\mathbb{P}_i\{\tau^{\mathrm{o}} \le n, T_n = m, X_n \in F\} &= \mathbb{P}_i\{X_1 \in E^\dagger, \tau^{\mathrm{o}} \le n, T_n = m, X_n \in F\} \\
&\quad + \mathbb{P}_i\{X_1 \in E^-, \tau^{\mathrm{o}} \le n, T_n = m, X_n \in F\} \\
&= \sum_{j\in E^{\mathrm{o}}}\sum_{\ell=1}^{n-1} \mathbb{P}_i\{X_1 \in E^\dagger, \tau^{\mathrm{o}} = \ell, X_{\tau^{\mathrm{o}}} = j\}\, \mathbb{P}_j\{T_{n-\ell} = m-\ell, X_{n-\ell} \in F\} \\
&\quad + \sum_{j\in E^{\mathrm{o}}}\sum_{\ell=2}^{n} \mathbb{P}_i\{X_1 \in E^-, \tau^{\mathrm{o}} = \ell, X_{\tau^{\mathrm{o}}} = j\}\, \mathbb{P}_j\{T_{n-\ell} = m-1, X_{n-\ell} \in F\}.
\end{aligned}
$$

The sum in (7.1) can be evaluated as follows:

$$
\sum_{\substack{m,n\in\mathbb{N}:\\ 1\le m\le n-1}} \mathbb{P}_i\{\tau^{\mathrm{o}} \le n, T_n = m, X_n \in F\}\, x^m y^{n-m}
$$

$$
\begin{aligned}
&= \sum_{\ell=1}^{\infty} \sum_{j\in E^{\mathrm{o}}} \mathbb{P}_i\{X_1 \in E^\dagger, \tau^{\mathrm{o}} = \ell, X_{\tau^{\mathrm{o}}} = j\} \sum_{\substack{m,n\in\mathbb{N}:\\ \ell\le m\le n-1}} \mathbb{P}_j\{T_{n-\ell} = m-\ell, X_{n-\ell} \in F\}\, x^m y^{n-m} \\
&\quad + \sum_{\ell=2}^{\infty} \sum_{j\in E^{\mathrm{o}}} \mathbb{P}_i\{X_1 \in E^-, \tau^{\mathrm{o}} = \ell, X_{\tau^{\mathrm{o}}} = j\} \sum_{\substack{m,n\in\mathbb{N}:\\ 1\le m\le n-\ell+1}} \mathbb{P}_j\{T_{n-\ell} = m-1, X_{n-\ell} \in F\}\, x^m y^{n-m} \\
&= \sum_{j\in E^{\mathrm{o}}} \sum_{\ell=1}^{\infty} \mathbb{P}_i\{X_1 \in E^\dagger, \tau^{\mathrm{o}} = \ell, X_{\tau^{\mathrm{o}}} = j\}\, x^\ell \sum_{\substack{m,n\in\mathbb{N}:\\ m\le n-1}} \mathbb{P}_j\{T_n = m, X_n \in F\}\, x^m y^{n-m} \\
&\quad + \sum_{j\in E^{\mathrm{o}}} \sum_{\ell=2}^{\infty} \mathbb{P}_i\{X_1 \in E^-, \tau^{\mathrm{o}} = \ell, X_{\tau^{\mathrm{o}}} = j\}\, x y^{\ell-1} \sum_{\substack{m,n\in\mathbb{N}:\\ m\le n}} \mathbb{P}_j\{T_n = m, X_n \in F\}\, x^m y^{n-m} \\
&= \sum_{j\in E^{\mathrm{o}}} \mathbb{E}_i\big(x^{\tau^{\mathrm{o}}} \mathbb{1}_{\{X_1\in E^\dagger, X_{\tau^{\mathrm{o}}}=j\}}\big)[K_j(x,y) - K_j(x,0)] + \frac{x}{y} \sum_{j\in E^{\mathrm{o}}} \mathbb{E}_i\big(y^{\tau^{\mathrm{o}}} \mathbb{1}_{\{X_1\in E^-, X_{\tau^{\mathrm{o}}}=j\}}\big) K_j(x,y) \\
&= \sum_{j\in E^{\mathrm{o}}} \left(H_{ij}^{\mathrm{o}\dagger}(x) + \frac{x}{y}\, H_{ij}^{\mathrm{o}-}(y)\right) K_j(x,y) - \sum_{j\in E^{\mathrm{o}}} H_{ij}^{\mathrm{o}\dagger}(x) K_j(x,0). \qquad (7.2)
\end{aligned}
$$

As a result, putting (7.2) into (7.1), we get (3.11). □

7.2 Proof of Proposition 3.1

By putting (3.12) into (3.11), we obtain, for all $i \in \mathcal{E}$, that

$$
\begin{aligned}
K_i(x,y) = G_i(x) &- \sum_{j\in E^-} H_{ij}^-(x) G_j(x) + G_i(y) - \sum_{j\in E^\dagger} H_{ij}^\dagger(y) G_j(y) - \sum_{j\in E^{\mathrm{o}}} H_{ij}^{\mathrm{o}\dagger}(x) G_j(x) - \mathbb{1}_F(i) \\
&+ \sum_{j\in E^{\mathrm{o}}} \left(H_{ij}^{\mathrm{o}\dagger}(x) + \frac{x}{y}\, H_{ij}^{\mathrm{o}-}(y)\right) K_j(x,y) + \sum_{k\in E^{\mathrm{o}}} H_{ik}^{\mathrm{o}\dagger}(x) \sum_{j\in E^-} H_{kj}^-(x) G_j(x). \qquad (7.3)
\end{aligned}
$$

If $i \in E^+$, we have $H_{ij}^\dagger(y) = p_{ij}\, y$, $H_{ij}^{\mathrm{o}\dagger}(x) = H_{ij}^{\mathrm{o}}(x)$, and $H_{ij}^{\mathrm{o}-}(y) = 0$. Putting this into (7.3), we get that

$$
\begin{aligned}
K_i(x,y) = G_i(x) &- \sum_{j\in E^{\mathrm{o}}} H_{ij}^{\mathrm{o}}(x) G_j(x) + \sum_{j\in E^{\mathrm{o}}} H_{ij}^{\mathrm{o}}(x) K_j(x,y) \\
&+ \sum_{j\in E^-} \left(\sum_{k\in E^{\mathrm{o}}} H_{ik}^{\mathrm{o}}(x) H_{kj}^-(x) - H_{ij}^-(x)\right) G_j(x) + \left(G_i(y) - y \sum_{j\in E^\dagger} p_{ij} G_j(y) - \mathbb{1}_F(i)\right). \qquad (7.4)
\end{aligned}
$$

We claim that both terms within brackets in the second line of (7.4) vanish. Indeed, for $j \in E^-$, since $\tau^{\mathrm{o}} < \tau^-$, we have that

$$
H_{ij}^-(x) = \mathbb{E}_i\big(x^{\tau^-} \mathbb{1}_{\{X_{\tau^-}=j\}}\big) = \sum_{k\in E^{\mathrm{o}}} \mathbb{E}_i\big(x^{\tau^{\mathrm{o}}} \mathbb{1}_{\{X_{\tau^{\mathrm{o}}}=k\}}\big)\, \mathbb{E}_k\big(x^{\tau^-} \mathbb{1}_{\{X_{\tau^-}=j\}}\big) = \sum_{k\in E^{\mathrm{o}}} H_{ik}^{\mathrm{o}}(x) H_{kj}^-(x).
$$

Therefore, the first term within brackets in the second line of (7.4) vanishes. Moreover, by (3.2), we have that

$$
G_i(y) = \sum_{k\in F} G_{ik}(y) = \sum_{k\in F} \left(\delta_{ik} + y \sum_{j\in E^\dagger} p_{ij} G_{jk}(y)\right) = y \sum_{j\in E^\dagger} p_{ij} G_j(y) + \mathbb{1}_F(i)
$$

which proves that the second term within brackets in (7.4) vanishes too. Hence we have checked that (3.13) holds true.

On the other hand, for $i \in E^-$, we have necessarily $\tau^{\mathrm{o}} = \tau^\dagger$ and then $H_{ij}^\dagger(y) = H_{ij}^{\mathrm{o}}(y)\, \mathbb{1}_{E^{\mathrm{o}}}(j)$. Moreover, if $i \in E^-$ and $X_1 \in E^\dagger$, then $\tau^{\mathrm{o}} = 1$ and $X_1 \in E^{\mathrm{o}}$ and we have that

$$
H_{ij}^{\mathrm{o}\dagger}(x) = p_{ij}\, x\, \mathbb{1}_{E^{\mathrm{o}}}(j), \quad H_{ij}^{\mathrm{o}-}(y) = \mathbb{E}_i\big(y^{\tau^{\mathrm{o}}} \mathbb{1}_{\{X_{\tau^{\mathrm{o}}}=j\}}\big) - \mathbb{E}_i\big(y^{\tau^{\mathrm{o}}} \mathbb{1}_{\{X_1\in E^\dagger, X_{\tau^{\mathrm{o}}}=j\}}\big) = H_{ij}^{\mathrm{o}}(y) - p_{ij}\, y\, \mathbb{1}_{E^{\mathrm{o}}}(j).
$$

Then, putting these equalities into (7.3), we get that

$$
\begin{aligned}
K_i(x,y) = G_i(y) &- \sum_{j\in E^{\mathrm{o}}} H_{ij}^{\mathrm{o}}(y) G_j(y) + \frac{x}{y} \sum_{j\in E^{\mathrm{o}}} H_{ij}^{\mathrm{o}}(y) K_j(x,y) \\
&+ \left(G_i(x) - x \sum_{j\in E^{\mathrm{o}}} p_{ij} G_j(x) - \mathbb{1}_F(i)\right) + \sum_{j\in E^-} \left(x \sum_{k\in E^{\mathrm{o}}} p_{ik} H_{kj}^-(x) - H_{ij}^-(x)\right) G_j(x). \qquad (7.5)
\end{aligned}
$$

We claim that the sum of both terms within brackets in the second line of (7.5) vanish. This can be easily seen by using (3.2) and Assumption (A_1) which yield that

$$G_i(x) - x \sum_{j\in E^{\circ}} p_{ij} G_j(x) - \mathbb{1}_F(i) = x \sum_{j\in E^-} p_{ij} G_j(x)$$

and by observing that

$$H^-_{ij}(x) = \mathbb{E}_i\big(x^{\tau^-}\mathbb{1}_{\{X_1\in E^-, X_{\tau^-}=j\}}\big) + \mathbb{E}_i\big(x^{\tau^-}\mathbb{1}_{\{X_1\in E^{\circ}, X_{\tau^-}=j\}}\big) = x \sum_{k\in E^{\circ}} p_{ik} H^-_{kj}(x) + p_{ij}\, x\, \mathbb{1}_{E^-}(j).$$

Finally, Equality (7.5) can be simplified to (3.14). □

7.3 Proof of Proposition 2.1

Set $A^{\circ}_{0,m} = \{X_m \in E^{\circ}\}$. We notice that for any $m \in \mathbb{N}^*$ and any $\ell \in \{1,\dots,m\}$, $A_{\ell,m} = A^{\circ}_{0,m} \cap A_{\ell-1,m-1}$. Therefore, putting this into the definition of B_m, we get, for any $m \in \mathbb{N}^*$, that

$$\begin{aligned} B_m &= A_{0,m} \cup A_{1,m} \cup \dots \cup A_{m-1,m} \\ &= A_{0,m} \cup \big(A^{\circ}_{0,m} \cap A_{0,m-1}\big) \cup \dots \cup \big(A^{\circ}_{0,m} \cap A_{m-2,m-1}\big) \\ &= A_{0,m} \cup \big[A^{\circ}_{0,m} \cap (A_{0,m-1} \cup \dots \cup A_{m-2,m-1})\big] \end{aligned}$$

and we see, for any $m \in \mathbb{N}\backslash\{0,1\}$, that

$$B_m = A_{0,m} \cup \big(A^{\circ}_{0,m} \cap B_{m-1}\big). \tag{7.6}$$

In the same way, we have that

$$B'_m = A'_{0,m} \cup \big(A^{\circ}_{0,m} \cap B'_{m-1}\big). \tag{7.7}$$

By observing that $\big(A^{\circ}_{0,m}\big)^c \cap A^c_{0,m} = A'_{0,m}$ and using the elementary equality $A \cup B = A \cup (B\backslash A)$, we obtain from (7.6) that

$$\begin{aligned} B^c_m &= A^c_{0,m} \cap \big(\big(A^{\circ}_{0,m}\big)^c \cup B^c_{m-1}\big) \\ &= \big(\big(A^{\circ}_{0,m}\big)^c \cap A^c_{0,m}\big) \cup \big(A^c_{0,m} \cap B^c_{m-1}\big) \\ &= A'_{0,m} \cup \big[\big(A^c_{0,m} \cap B^c_{m-1}\big) \backslash A'_{0,m}\big]. \end{aligned}$$

The term within brackets in the foregoing equality can be simplified as follows:

$$\big(A^c_{0,m} \cap B^c_{m-1}\big) \backslash A'_{0,m} = \big(A^c_{0,m} \backslash A'_{0,m}\big) \cap B^c_{m-1} = A^{\circ}_{0,m} \cap B^c_{m-1}.$$

As a by-product, for any $m \in \mathbb{N}^*$,

$$B^c_m = A'_{0,m} \cup \big(A^{\circ}_{0,m} \cap B^c_{m-1}\big). \tag{7.8}$$

Now, in view of (7.7) and (7.8), it is clear by recurrence that $B^c_m = B'_m$ as claimed. □

7.4 Proof of Theorem 3.2

The observations below will be useful to compute the generating function of $\tilde{T}_n$.

Lemma 7.1 *Assume that $X_0 \in E^{\circ}$. The following hold true: for any $n \in \mathbb{N}^*$,*

- *$\tilde{T}_n = 0$ if and only if $\tau^+ > n$;*
- *$\tilde{T}_n = n$ if and only if $X_1 \in E^+$ and $\tau^- > n$;*
- *if $1 \le \tilde{T}_n \le n-1$, then $\tilde{\tau}^{\pm} \le n$ and $\tilde{T}_n = (\tilde{\tau}^{\pm} - 1)\mathbb{1}_{E^+} + \tilde{T}'_n$ where $\tilde{T}'_n = \sum_{m=\tilde{\tau}^{\pm}}^{n} \delta_m$. Additionally, $\tilde{T}'_n$ has the same distribution as $\tilde{T}_{n-\tilde{\tau}^{\pm}+1}$ and is independent from $X_1,\dots,X_{\tilde{\tau}^{\pm}-1}$.*

Proof
Fix a positive integer n.

- Concerning the first point, we have that
  - the set $\{X_1,\dots,X_m \in E^- \cup E^{\circ}\}$ is included in B'_m for all $m \in \{1,\dots,n\}$. Then, if $X_m \in E^- \cup E^{\circ}$ for all $m \in \{1,\dots,n\}$, we have $\delta_1 = \dots = \delta_n = 0$ and $\tilde{T}_n = 0$;

- if there existed an $\ell \in \{1, \dots, n\}$ such that $X_\ell \in E^+$, we would have of course $\tilde{T}_n \geq 1$.

This is equivalent to saying that $\tau^+ > n$.

- Concerning the second point, we have that
  - the set $\{X_1 \in E^+, X_2, \dots, X_m \in E^+ \cup E^o\}$ is included in B_m for all $m \in \{1, \dots, n\}$. Then, if $X_1 \in E^+$ and $X_2, \dots, X_n \in E^+ \cup E^o$, we have $\delta_1 = \dots = \delta_n = 1$ and $\tilde{T}_n = n$;
  - if there existed an $\ell \in \{2, \dots, n\}$ such that $X_\ell \in E^-$, we would have of course $\tilde{T}_n \leq n-1$. If we had $X_1 \in E^- \cup E^o$, we would have $\delta_1 = 0$ and $\tilde{T}_n \leq n-1$ too.

  This is equivalent to saying that $X_1 \in E^+$ and $\tau^- > n$.

- Concerning the third point, we have that
  - $1 \leq \tilde{T}_n \leq n-1$ if and only if there exists distinct integers $\ell, \ell' \in \{1, \dots, n\}$ such that $\delta_\ell = 1$ and $\delta_{\ell'} = 0$. This implies that there exists necessarily an integer $m \in \{1, \dots, n\}$ such that $X_m \in E^o$. Else, because of Assumptions (A_1) and (A_2), we would have $X_m \in E^+$ for all $m \in \{1, \dots, n\}$ or $X_m \in E^-$ for all $m \in \{1, \dots, n\}$; in both cases, we would have $\tilde{T}_n \in \{0, n\}$. Therefore $1 \leq \tau^o \leq n$.
  - On the other hand, we must take care of the successive points lying in E^o after time τ^o. For describing them, time $\tilde{\tau}^\pm$ will be useful. Obviously, we have $\tilde{\tau}^\pm > \tau^o \geq 1$ and between times τ^o and $\tilde{\tau}^\pm - 1$, the chain $(X_n)_{n\in\mathbb{N}}$ stays in E^o. More precisely, we must take care of the points corresponding to the times between τ^o and $(\tilde{\tau}^\pm - 1) \wedge n$. In fact, these $(\tilde{\tau}^\pm - 1) \wedge n - \tau^o$ points are counted in $\tilde{T}_n$ if $X_1 \in E^+$, not counted if $X_1 \in E^- \cup E^o$. Then, if $\tilde{\tau}^\pm \leq n$, we have $\tilde{T}_n = \tilde{T}_{\tilde{\tau}^\pm - 1} + \tilde{T}'_n$ with

$$\tilde{T}_{\tilde{\tau}^\pm - 1} = \begin{cases} \tilde{\tau}^\pm - 1 & \text{if } X_1 \in E^+, \\ 0 & \text{if } X_1 \in E^- \cup E^o, \end{cases} \quad \text{and} \quad \tilde{T}'_n = \sum_{m=\tilde{\tau}^\pm}^{n} \delta_m.$$

□

Now, we prove Theorem 3.2. The points displayed in Lemma 7.1 entail that, for $i \in E^o$ and for $n \in \mathbb{N}^*$,

$$\begin{aligned} \mathbb{P}_i\{X_1 \in E^\pm, \tilde{T}_n = 0, X_n \in F\} &= \mathbb{P}_i\{X_1 \in E^-, \tau^+ > n, X_n \in F\} \\ &= \sum_{j \in E^-} p_{ij}\, \mathbb{P}_j\{\tau^+ > n-1, X_{n-1} \in F\}, \\ \mathbb{P}_i\{X_1 \in E^\pm, \tilde{T}_n = n, X_n \in F\} &= \mathbb{P}_i\{X_1 \in E^+, \tau^- > n, X_n \in F\} \\ &= \sum_{j \in E^+} p_{ij}\, \mathbb{P}_j\{\tau^- > n-1, X_{n-1} \in F\}, \end{aligned} \tag{7.9}$$

and for $i \in E^o$ and $m \in \{1, \dots, n-1\}$,

$$\mathbb{P}_i\{X_1 \in E^\pm, \tilde{T}_n = m, X_n \in F\} = \sum_{\ell=2}^{n} \sum_{j \in E^o} \mathbb{P}_i\{X_1 \in E^\pm, \tilde{\tau}^\pm = \ell, X_{\tilde{\tau}^\pm - 1} = j, \tilde{T}_n = m, X_n \in F\} \tag{7.10}$$

where, for $\ell \in \{2, \dots, n\}$,

$$\begin{aligned} &\mathbb{P}_i\{X_1 \in E^\pm, \tilde{\tau}^\pm = \ell, X_{\tilde{\tau}^\pm - 1} = j, \tilde{T}_n = m, X_n \in F\} \\ &\quad = \mathbb{P}_i\{X_1 \in E^+, \tilde{\tau}^\pm = \ell, X_{\tilde{\tau}^\pm - 1} = j\}\, \mathbb{P}_j\{X_1 \in E^\pm, \tilde{T}_{n-\ell+1} = m - \ell + 1, X_{n-\ell+1} \in F\} \\ &\qquad + \mathbb{P}_i\{X_1 \in E^-, \tilde{\tau}^\pm = \ell, X_{\tilde{\tau}^\pm - 1} = j\}\, \mathbb{P}_j\{X_1 \in E^\pm, \tilde{T}_{n-\ell+1} = m, X_{n-\ell+1} \in F\}. \end{aligned}$$

By (7.9) and by imitating the proof of (3.12), it can be easily seen that $\tilde{K}_i(x, 0)$ and $\tilde{K}_i(0, y)$ can be written as (3.20). Next, we compute $\tilde{K}_i(x, y)$. By (7.10),

$$\begin{aligned} \tilde{K}_i(x, y) &= \tilde{K}_i(x, 0) + \tilde{K}_i(0, y) - \mathbb{P}_i\{X_0 \in F, X_1 \in E^\pm\} + \sum_{\substack{m,n \in \mathbb{N}: \\ 1 \leq m \leq n-1}} \mathbb{P}_i\{X_1 \in E^\pm, \tilde{T}_n = m, X_n \in F\}\, x^m y^{n-m} \\ &= \tilde{K}_i(x, 0) + \tilde{K}_i(0, y) - \mathbb{1}_F(i) \mathbb{P}_i\{X_1 \in E^\pm\} \\ &\quad + \sum_{\ell=2}^{\infty} \sum_{j \in E^o} \mathbb{P}_i\{X_1 \in E^+, \tilde{\tau}^\pm = \ell, X_{\tilde{\tau}^\pm - 1} = j\} \end{aligned}$$

$$
\begin{aligned}
&\times \sum_{\substack{m,n\in\mathbb{N}:\\ \ell-1\le m\le n-1}} \mathbb{P}_j\{X_1\in E^{\pm},\tilde{T}_{n-\ell+1}=m-\ell+1, X_{n-\ell+1}\in F\}\, x^m y^{n-m}\\
&+\sum_{\ell=2}^{\infty}\sum_{j\in E^{\circ}} \mathbb{P}_i\{X_1\in E^-,\tilde{\tau}^{\pm}=\ell, X_{\tilde{\tau}^{\pm}-1}=j\}\\
&\times \sum_{\substack{m,n\in\mathbb{N}:\\ 1\le m\le n-\ell+1}} \mathbb{P}_j\{X_1\in E^{\pm},\tilde{T}_{n-\ell+1}=m, X_{n-\ell+1}\in F\}\, x^m y^{n-m}\\
={}&\tilde{K}_i(x,0)+\tilde{K}_i(0,y)-\mathbb{1}_F(i)\mathbb{P}_i\{X_1\in E^{\pm}\}\\
&+\sum_{\ell=2}^{\infty}\sum_{j\in E^{\circ}} \mathbb{P}_i\{X_1\in E^+,\tilde{\tau}^{\pm}=\ell, X_{\tilde{\tau}^{\pm}-1}=j\}\, x^{\ell-1}\big(\tilde{K}_j(x,y)-\tilde{K}_j(x,0)\big)\\
&+\sum_{\ell=2}^{\infty}\sum_{j\in E^{\circ}} \mathbb{P}_i\{X_1\in E^-,\tilde{\tau}^{\pm}=\ell, X_{\tilde{\tau}^{\pm}-1}=j\}\, y^{\ell-1}\big(\tilde{K}_j(x,y)-\tilde{K}_j(0,y)\big)\\
={}&\tilde{K}_i(x,0)+\tilde{K}_i(0,y)+\sum_{j\in E^{\circ}}\big(\tilde{H}_{ij}^{\pm+}(x)+\tilde{H}_{ij}^{\pm-}(y)\big)\tilde{K}_j(x,y)\\
&-\sum_{j\in E^{\circ}}\tilde{H}_{ij}^{\pm+}(x)\tilde{K}_j(x,0)-\sum_{j\in E^{\circ}}\tilde{H}_{ij}^{\pm-}(y)\tilde{K}_j(0,y)-\mathbb{1}_F(i)\mathbb{P}_i\{X_1\in E^{\pm}\},
\end{aligned}
$$

as claimed. □

7.5 Proof of Theorem 6.1

- *First step: writing out the matrices*

  Suppose $F=\mathbb{Z}$ and set $\varpi=\pi_1+2\pi_2$, $\mathbf{1}=\begin{pmatrix}1\\1\end{pmatrix}$, $\mathbb{1}=\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$. Below, we rewrite the matrices displayed in Example 4.1 which can be simplified under the condition of symmetry $\pi_{-1}=\pi_1$, $\pi_{-2}=\pi_2$ and $\Gamma_{-j}=\Gamma_j$ for any integer j:

$$\mathbf{I}=\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad \mathbf{I}^{\pm}=\varpi\mathbf{I},\quad \mathbf{1_F}=\mathbf{1},\quad \mathbf{1_F^{\pm}}=\varpi\mathbf{1},$$

$$\mathbf{P}=\begin{pmatrix}\pi_0&\pi_1\\\pi_1&\pi_0\end{pmatrix},\quad \mathbf{P}_-=\begin{pmatrix}\pi_2&\pi_1\\0&\pi_2\end{pmatrix},\quad \mathbf{P}_+=\begin{pmatrix}\pi_2&0\\\pi_1&\pi_2\end{pmatrix},\quad \mathbf{P}_\dagger=\begin{pmatrix}\pi_0&\pi_1&\pi_2&0\\\pi_1&\pi_0&\pi_1&\pi_2\end{pmatrix},$$

$$\mathbf{\Gamma}(x)=\begin{pmatrix}\Gamma_0(x)&\Gamma_1(x)\\\Gamma_1(x)&\Gamma_0(x)\end{pmatrix},\quad \mathbf{\Gamma}_-(x)=\begin{pmatrix}\Gamma_2(x)&\Gamma_1(x)\\\Gamma_3(x)&\Gamma_2(x)\end{pmatrix},\quad \mathbf{\Gamma}_+(x)=\begin{pmatrix}\Gamma_2(x)&\Gamma_3(x)\\\Gamma_1(x)&\Gamma_2(x)\end{pmatrix},$$

$$\mathbf{\Gamma}_=(x)=\begin{pmatrix}\Gamma_4(x)&\Gamma_3(x)\\\Gamma_5(x)&\Gamma_4(x)\end{pmatrix},\quad \mathbf{\Gamma}_\ddagger(x)=\begin{pmatrix}\Gamma_4(x)&\Gamma_5(x)\\\Gamma_3(x)&\Gamma_4(x)\end{pmatrix},$$

$$\mathbf{G}(x)=\mathbf{G}_+(x)=\mathbf{G}_-(x)=\frac{1}{1-x}\mathbf{1},\quad \mathbf{G}_\dagger(x)=\frac{1}{1-x}\mathbb{1},$$

$$\mathbf{K}(x,y)=\begin{pmatrix}K_0(x,y)\\K_1(x,y)\end{pmatrix},\quad \tilde{\mathbf{K}}(x,y)=\begin{pmatrix}\tilde{K}_0(x,y)\\\tilde{K}_1(x,y)\end{pmatrix}.$$

  We observe that $\mathbf{P}$ and $\mathbf{\Gamma}(x)$ are symmetric and that $\mathbf{P}_+$, $\mathbf{\Gamma}_+(x)$, $\mathbf{\Gamma}_\ddagger(x)$ are the transposes of $\mathbf{P}_-$, $\mathbf{\Gamma}_-(x)$, $\mathbf{\Gamma}_=(x)$, respectively. Let us write the inverses of $\mathbf{\Gamma}(x)$ and $\mathbf{I}-x\,\mathbf{P}$. For this, we introduce their respective determinants:

$$\delta(x)=(\pi_0^2-\pi_1^2)x^2-2\pi_0x+1,\quad \Delta(x)=\Gamma_0(x)^2-\Gamma_1(x)^2.$$

  We have

$$\mathbf{\Gamma}(x)^{-1}=\frac{1}{\Delta(x)}\begin{pmatrix}\Gamma_0(x)&-\Gamma_1(x)\\-\Gamma_1(x)&\Gamma_0(x)\end{pmatrix},\quad (\mathbf{I}-x\,\mathbf{P})^{-1}=\frac{1}{\delta(x)}\begin{pmatrix}1-\pi_0x&\pi_1x\\\pi_1x&1-\pi_0x\end{pmatrix}.$$

- *Second step: deriving the generating matrix* $\mathbf{K}$

  In view of (4.15), we write $\mathbf{K}(x,y)$ as $\mathbf{K}(x,y)=\mathbf{D}(x,y)^{-1}\mathbf{N}(x,y)$ where

$$\mathbf{D}(x,y)=\mathbf{I}-x\,(\mathbf{P}+\mathbf{D}_+(x)+\mathbf{D}_-(y))$$

$$\mathbf{N}(x,y)=\frac{1}{1-x}\big[\mathbf{I}-x\big(\mathbf{P}+\mathbf{D}_{+}(x)\big)\big]\mathbf{N}_{-}(x)+\frac{1}{1-y}\mathbf{N}_{\dagger}(y)-\mathbf{1}$$

with

$$\mathbf{D}_{+}(x)=\mathbf{P}_{+}\boldsymbol{\Gamma}_{-}(x)\boldsymbol{\Gamma}(x)^{-1},\quad \mathbf{D}_{-}(y)=\mathbf{P}_{-}\boldsymbol{\Gamma}_{+}(y)\boldsymbol{\Gamma}(y)^{-1},$$
$$\mathbf{N}_{-}(x)=\mathbf{1}-\boldsymbol{\Gamma}_{-}(x)\boldsymbol{\Gamma}(x)^{-1}\mathbf{1},\quad \mathbf{N}_{\dagger}(y)=\mathbf{1}-y\,\mathbf{P}_{\dagger}\mathbb{1}-y\,\mathbf{D}_{-}(y)\mathbf{1}.$$

First, we calculate

$$\mathbf{D}_{+}(x)=\frac{1}{\Delta(x)}\begin{pmatrix}\pi_2 & 0\\ \pi_1 & \pi_2\end{pmatrix}\begin{pmatrix}\Gamma_2(x) & \Gamma_1(x)\\ \Gamma_3(x) & \Gamma_2(x)\end{pmatrix}\begin{pmatrix}\Gamma_0(x) & -\Gamma_1(x)\\ -\Gamma_1(x) & \Gamma_0(x)\end{pmatrix}=\frac{1}{\Delta(x)}\begin{pmatrix}D_{00}^{+}(x) & D_{01}^{+}(x)\\ D_{10}^{+}(x) & D_{11}^{+}(x)\end{pmatrix}$$

with

$$\begin{aligned}
D_{00}^{+}(x)&=\pi_2\big(\Gamma_0(x)\Gamma_2(x)-\Gamma_1(x)^2\big)\\
D_{01}^{+}(x)&=\pi_2\big(\Gamma_0(x)\Gamma_1(x)-\Gamma_1(x)\Gamma_2(x)\big),\\
D_{10}^{+}(x)&=\pi_1\big(\Gamma_0(x)\Gamma_2(x)-\Gamma_1(x)^2\big)+\pi_2\big(\Gamma_0(x)\Gamma_3(x)-\Gamma_1(x)\Gamma_2(x)\big),\\
D_{11}^{+}(x)&=\pi_1\big(\Gamma_0(x)\Gamma_1(x)-\Gamma_1(x)\Gamma_2(x)\big)+\pi_2\big(\Gamma_0(x)\Gamma_2(x)-\Gamma_1(x)\Gamma_3(x)\big).
\end{aligned}$$

Similarly,

$$\mathbf{D}_{-}(y)=\frac{1}{\Delta(y)}\begin{pmatrix}\pi_2 & \pi_1\\ 0 & \pi_2\end{pmatrix}\begin{pmatrix}\Gamma_2(y) & \Gamma_3(y)\\ \Gamma_1(y) & \Gamma_2(y)\end{pmatrix}\begin{pmatrix}\Gamma_0(y) & -\Gamma_1(y)\\ -\Gamma_1(y) & \Gamma_0(y)\end{pmatrix}=\frac{1}{\Delta(y)}\begin{pmatrix}D_{00}^{-}(y) & D_{01}^{-}(y)\\ D_{10}^{-}(y) & D_{11}^{-}(y)\end{pmatrix}$$

with

$$\begin{aligned}
D_{00}^{-}(y)&=\pi_1\big(\Gamma_0(y)\Gamma_1(y)-\Gamma_1(y)\Gamma_2(y)\big)+\pi_2\big(\Gamma_0(y)\Gamma_2(y)-\Gamma_1(y)\Gamma_3(y)\big),\\
D_{01}^{-}(y)&=\pi_1\big(\Gamma_0(y)\Gamma_2(y)-\Gamma_1(y)^2\big)+\pi_2\big(\Gamma_0(y)\Gamma_3(y)-\Gamma_1(y)\Gamma_2(y)\big),\\
D_{10}^{-}(y)&=\pi_2\big(\Gamma_0(y)\Gamma_1(y)-\Gamma_1(y)\Gamma_2(y)\big),\\
D_{11}^{-}(y)&=\pi_2\big(\Gamma_0(y)\Gamma_2(y)-\Gamma_1(y)^2\big).
\end{aligned}$$

With this at hand,

$$\mathbf{D}(x,y)^{-1}=\frac{\Delta(x)\Delta(y)}{D(x,y)}\begin{pmatrix}(1-\pi_0x)\Delta(x)\Delta(y)-x\Delta(y)D_{11}^{+}(x)-y\Delta(x)D_{11}^{-}(y) & \pi_1x\Delta(x)\Delta(y)+x\Delta(y)D_{01}^{+}(x)+y\Delta(x)D_{01}^{-}(y)\\ \pi_1x\Delta(x)\Delta(y)+x\Delta(y)D_{10}^{+}(x)+y\Delta(x)D_{10}^{-}(y) & (1-\pi_0x)\Delta(x)\Delta(y)-x\Delta(y)D_{00}^{+}(x)-y\Delta(x)D_{00}^{-}(y)\end{pmatrix}\tag{7.11}$$

where $D(x,y)$ is the determinant of the previous displayed matrix.
On the other hand, we have

$$\mathbf{N}_{-}(x)=\begin{pmatrix}1\\1\end{pmatrix}-\frac{1}{\Delta(x)}\begin{pmatrix}\Gamma_2(x) & \Gamma_1(x)\\ \Gamma_3(x) & \Gamma_2(x)\end{pmatrix}\begin{pmatrix}\Gamma_0(x) & -\Gamma_1(x)\\ -\Gamma_1(x) & \Gamma_0(x)\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}=\frac{1}{\Delta(x)}\begin{pmatrix}N_0^{-}(x)\\ N_1^{-}(x)\end{pmatrix}$$

with

$$\begin{aligned}
N_0^{-}(x)&=\Delta(x)-\big(\Gamma_0(x)-\Gamma_1(x)\big)\big(\Gamma_1(x)+\Gamma_2(x)\big),\\
N_1^{-}(x)&=\Delta(x)-\big(\Gamma_0(x)-\Gamma_1(x)\big)\big(\Gamma_2(x)+\Gamma_3(x)\big).
\end{aligned}$$

Next, we calculate $\mathbf{N}_{\dagger}(y)$:

$$\mathbf{N}_{\dagger}(y)=\begin{pmatrix}1\\1\end{pmatrix}-y\begin{pmatrix}\pi_0 & \pi_1 & \pi_2 & 0\\ \pi_1 & \pi_0 & \pi_1 & \pi_2\end{pmatrix}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}-\frac{y}{\Delta(y)}\begin{pmatrix}D_{00}^{-}(y) & D_{01}^{-}(y)\\ D_{10}^{-}(y) & D_{11}^{-}(y)\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}=\frac{1}{\Delta(y)}\begin{pmatrix}N_0^{\dagger}(y)\\ N_1^{\dagger}(y)\end{pmatrix}$$

with

$$\begin{aligned}
N_0^{\dagger}(y)&=\big(1-(1-\pi_1-\pi_2)y\big)\Delta(y)-yD_{00}^{-}(y)-yD_{01}^{-}(y)\\
&=\big(1-(1-\pi_1-\pi_2)y\big)\Delta(y)-y\big(\Gamma_0(y)-\Gamma_1(y)\big)\big(\pi_1\Gamma_1(y)+(\pi_1+\pi_2)\Gamma_2(y)+\pi_2\Gamma_3(y)\big),\\
N_1^{\dagger}(y)&=\big(1-(1-\pi_2)y\big)\Delta(y)-yD_{10}^{-}(y)-yD_{11}^{-}(y)\\
&=\big(1-(1-\pi_2)y\big)\Delta(y)-\pi_2y\big(\Gamma_0(y)-\Gamma_1(y)\big)\big(\Gamma_1(y)+\Gamma_2(y)\big).
\end{aligned}$$

As a by-product, we obtain the following representation:

$$\mathbf{N}(x,y) = \begin{pmatrix} N_0(x,y) \\ N_1(x,y) \end{pmatrix} \tag{7.12}$$

with

$$\begin{aligned}
N_0(x,y) &= \frac{1}{(1-x)\Delta(x)^2}\left[\big((1-\pi_0 x)\Delta(x) - xD^+_{00}(x)\big)N^-_0(x) - x\big(\pi_1\Delta(x) + D^+_{01}(x)\big)N^-_1(x)\right] \\
&\quad + \frac{1}{(1-y)\Delta(y)}\,N^\dagger_0(y) - 1, \\
N_1(x,y) &= \frac{1}{(1-x)\Delta(x)^2}\left[-x\big(\pi_1\Delta(x) + D^+_{10}(x)\big)N^-_0(x) + \big((1-\pi_0 x)\Delta(x) - xD^+_{11}(x)\big)N^-_1(x)\right] \\
&\quad + \frac{1}{(1-y)\Delta(y)}\,N^\dagger_1(y) - 1.
\end{aligned}$$

- *Third step: deriving the generating matrix* $\tilde{\mathbf{K}}$

In view of (4.16), we write $\tilde{\mathbf{K}}(x,y)$ as $\tilde{\mathbf{K}}(x,y) = \tilde{\mathbf{D}}(x,y)^{-1}\tilde{\mathbf{N}}(x,y)$ where

$$\begin{aligned}
\tilde{\mathbf{D}}(x,y) &= \mathbf{I} - \varpi x\,\tilde{\mathbf{D}}_+(x) - \varpi y\,\tilde{\mathbf{D}}_-(y), \\
\tilde{\mathbf{N}}(x,y) &= \big[\mathbf{I} - \varpi x\,\tilde{\mathbf{D}}_+(x)\big]\Big[\varpi\mathbf{1} + \frac{x}{1-x}\,\tilde{\mathbf{N}}_+(x)\Big] + \big[\mathbf{I} - \varpi y\,\tilde{\mathbf{D}}_-(y)\big]\Big[\varpi\mathbf{1} + \frac{y}{1-y}\,\tilde{\mathbf{N}}_-(y)\Big] - \varpi\mathbf{1},
\end{aligned}$$

with

$$\begin{gathered}
\tilde{\mathbf{D}}_+(x) = \mathbf{D}_+(x)(\mathbf{I} - x\,\mathbf{P})^{-1}, \quad \tilde{\mathbf{D}}_-(y) = \mathbf{D}_-(y)(\mathbf{I} - y\,\mathbf{P})^{-1}, \\
\tilde{\mathbf{N}}_+(x) = \mathbf{P}_+\big(\mathbf{1} - \mathbf{\Gamma}_=(x)\mathbf{\Gamma}(x)^{-1}\mathbf{1}\big), \quad \tilde{\mathbf{N}}_-(y) = \mathbf{P}_-\big(\mathbf{1} - \mathbf{\Gamma}_\ddagger(y)\mathbf{\Gamma}(y)^{-1}\mathbf{1}\big).
\end{gathered}$$

For instance,

$$\tilde{\mathbf{D}}_+(x) = \frac{1}{\delta(x)\Delta(x)}\,\mathbf{D}_+(x)\begin{pmatrix} 1-\pi_0 x & \pi_1 x \\ \pi_1 x & 1-\pi_0 x \end{pmatrix} = \frac{1}{\delta(x)\Delta(x)}\begin{pmatrix} \tilde{D}^+_{00}(x) & \tilde{D}^+_{01}(x) \\ \tilde{D}^+_{10}(x) & \tilde{D}^+_{11}(x) \end{pmatrix}$$

with

$$\begin{aligned}
\tilde{D}^+_{00}(x) &= \pi_2\big[\pi_1 x\,\Gamma_0(x)\Gamma_1(x) + (1-\pi_0 x)\Gamma_0(x)\Gamma_2(x) - (1-\pi_0 x)\Gamma_1(x)^2 - \pi_1 x\,\Gamma_1(x)\Gamma_2(x)\big], \\
\tilde{D}^+_{01}(x) &= \pi_2\big[(1-\pi_0 x)\Gamma_0(x)\Gamma_1(x) + \pi_2 x\Gamma_0(x)\Gamma_2(x) - \pi_1 x\,\Gamma_1(x)^2 - (1-\pi_0 x)\Gamma_1(x)\Gamma_2(x)\big], \\
\tilde{D}^+_{10}(x) &= \pi_1^2 x\,\Gamma_0(x)\Gamma_1(x) + \pi_1\big(1 + (\pi_2-\pi_0)x\big)\Gamma_0(x)\Gamma_2(x) + \pi_2(1-\pi_0 x)\Gamma_0(x)\Gamma_3(x) \\
&\quad - \pi_1(1-\pi_0 x)\Gamma_1(x)^2 - \big(\pi_2 + (\pi_1^2 - \pi_0\pi_2)x\big)\Gamma_1(x)\Gamma_2(x) - \pi_1\pi_2 x\,\Gamma_1(x)\Gamma_3(x), \\
\tilde{D}^+_{11}(x) &= \pi_1(1-\pi_0 x)\Gamma_0(x)\Gamma_1(x) + \big(\pi_2 + (\pi_1^2 - \pi_0\pi_2)x\big)\Gamma_0(x)\Gamma_2(x) + \pi_1\pi_2 x\,\Gamma_0(x)\Gamma_3(x) \\
&\quad - \pi_1^2 x\,\Gamma_1(x)^2 - \pi_1\big(1 + (\pi_2-\pi_0)x\big)\Gamma_1(x)\Gamma_2(x) - \pi_2(1-\pi_0 x)\Gamma_1(x)\Gamma_3(x).
\end{aligned}$$

Similarly, we obtain that

$$\tilde{\mathbf{D}}_-(y) = \frac{1}{\delta(y)\Delta(y)}\begin{pmatrix} \tilde{D}^-_{00}(y) & \tilde{D}^-_{01}(y) \\ \tilde{D}^-_{10}(y) & \tilde{D}^-_{11}(y) \end{pmatrix}$$

with

$$\begin{aligned}
\tilde{D}^-_{00}(y) &= \pi_1(1-\pi_0 y)\Gamma_0(y)\Gamma_1(y) + \big(\pi_2 + (\pi_1^2 - \pi_0\pi_2)y\big)\Gamma_0(y)\Gamma_2(y) + \pi_1\pi_2 y\,\Gamma_0(y)\Gamma_3(y) \\
&\quad - \pi_1^2 y\,\Gamma_1(y)^2 - \pi_1\big(1 + (\pi_2-\pi_0)y\big)\Gamma_1(y)\Gamma_2(y) - \pi_2(1-\pi_0 y)\Gamma_1(y)\Gamma_3(y), \\
\tilde{D}^-_{01}(y) &= \pi_1^2 y\,\Gamma_0(y)\Gamma_1(y) + \pi_1\big(1 + (\pi_2-\pi_0)y\big)\Gamma_0(y)\Gamma_2(y) + \pi_2(1-\pi_0 y)\Gamma_0(y)\Gamma_3(y) \\
&\quad - \pi_1(1-\pi_0 y)\Gamma_1(y)^2 - \big(\pi_2 + (\pi_1^2 - \pi_0\pi_2)y\big)\Gamma_1(y)\Gamma_2(y) - \pi_1\pi_2 y\,\Gamma_1(y)\Gamma_3(y), \\
\tilde{D}^-_{10}(y) &= \pi_2\big[(1-\pi_0 y)\Gamma_0(y)\Gamma_1(y) + \pi_1 y\,\Gamma_0(y)\Gamma_2(y) - \pi_1 y\,\Gamma_1(y)^2 - (1-\pi_0 y)\Gamma_1(y)\Gamma_2(y)\big], \\
\tilde{D}^-_{11}(y) &= \pi_2\big[\pi_1 y\,\Gamma_0(y)\Gamma_1(y) + (1-\pi_0 y)\Gamma_0(y)\Gamma_2(y) - (1-\pi_0 y)\Gamma_1(y)^2 - \pi_1 y\,\Gamma_1(y)\Gamma_2(y)\big].
\end{aligned}$$

With this at hand,

$$\tilde{\mathbf{D}}(x,y)^{-1} = \frac{\delta(x)\delta(y)\Delta(x)\Delta(y)}{\tilde{D}(x,y)}\begin{pmatrix} \delta(x)\delta(y)\Delta(x)\Delta(y) - \varpi x\delta(y)\Delta(y)\tilde{D}^+_{00}(x) - \varpi y\delta(x)\Delta(x)\tilde{D}^-_{00}(y) & \varpi x\delta(y)\Delta(y)\tilde{D}^+_{10}(x) + \varpi y\delta(x)\Delta(x)\tilde{D}^-_{10}(y) \\ \varpi x\delta(y)\Delta(y)\tilde{D}^+_{01}(x) + \varpi y\delta(x)\Delta(x)\tilde{D}^-_{01}(y) & \delta(x)\delta(y)\Delta(x)\Delta(y) - \varpi x\delta(y)\Delta(y)\tilde{D}^+_{11}(x) - \varpi y\delta(x)\Delta(x)\tilde{D}^-_{11}(y) \end{pmatrix} \tag{7.13}$$

where $\tilde{D}(x,y)$ is the determinant of the previous displayed matrix.

Concerning $\tilde{\mathbf{N}}(x)$,

$$\tilde{\mathbf{N}}_{+}(x) = \begin{pmatrix} \pi_2 & 0 \\ \pi_1 & \pi_2 \end{pmatrix}\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{1}{\Delta(x)}\begin{pmatrix} \Gamma_4(x) & \Gamma_3(x) \\ \Gamma_5(x) & \Gamma_4(x) \end{pmatrix}\begin{pmatrix} \Gamma_0(x) & -\Gamma_1(x) \\ -\Gamma_1(x) & \Gamma_0(x) \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right] = \frac{1}{\Delta(x)}\begin{pmatrix} \tilde{N}_0^+(x) \\ \tilde{N}_1^+(x) \end{pmatrix}$$

with

$$\begin{aligned}
\tilde{N}_0^+(x) &= \pi_2\Delta(x) - \pi_2\big(\Gamma_0(x) - \Gamma_1(x)\big)\big(\Gamma_3(x) + \Gamma_4(x)\big), \\
\tilde{N}_1^+(x) &= (\pi_1 + \pi_2)\Delta(x) - \big(\Gamma_0(x) - \Gamma_1(x)\big)\big(\pi_1\Gamma_3(x) + (\pi_1 + \pi_2)\Gamma_4(x) + \pi_2\Gamma_5(x)\big).
\end{aligned}$$

Similarly,

$$\tilde{\mathbf{N}}_{-}(y) = \frac{1}{\Delta(y)}\begin{pmatrix} \tilde{N}_0^-(y) \\ \tilde{N}_1^-(y) \end{pmatrix}$$

with

$$\begin{aligned}
\tilde{N}_0^-(y) &= (\pi_1 + \pi_2)\Delta(y) - \big(\Gamma_0(y) - \Gamma_1(y)\big)\big(\pi_1\Gamma_3(y) + (\pi_1 + \pi_2)\Gamma_4(y) + \pi_2\Gamma_5(y)\big), \\
\tilde{N}_1^-(y) &= \pi_2\Delta(y) - \pi_2\big(\Gamma_0(y) - \Gamma_1(y)\big)\big(\Gamma_3(y) + \Gamma_4(y)\big).
\end{aligned}$$

As a by-product, we obtain the following representation:

$$\tilde{\mathbf{N}}(x, y) = \begin{pmatrix} \tilde{N}_0(x, y) \\ \tilde{N}_1(x, y) \end{pmatrix} \tag{7.14}$$

with

$$\begin{aligned}
\tilde{N}_0(x, y) = {} & \left[1 - \frac{\varpi x}{\delta(x)\Delta(x)}\,\tilde{D}_{00}^+(x)\right]\left[\varpi + \frac{x}{(1-x)\Delta(x)}\,\tilde{N}_0^+(x)\right] \\
& - \frac{\varpi x}{\delta(x)\Delta(x)}\,\tilde{D}_{01}^+(x)\left[\varpi + \frac{x}{(1-x)\Delta(x)}\,\tilde{N}_1^+(x)\right] \\
& + \left[1 - \frac{\varpi y}{\delta(y)\Delta(y)}\,\tilde{D}_{00}^-(y)\right]\left[\varpi + \frac{y}{(1-y)\Delta(y)}\,\tilde{N}_0^-(y)\right] \\
& - \frac{\varpi y}{\delta(y)\Delta(y)}\,\tilde{D}_{01}^-(y)\left[\varpi + \frac{y}{(1-y)\Delta(y)}\,\tilde{N}_1^-(y)\right] - \varpi, \\
\tilde{N}_1(x, y) = {} & - \frac{\varpi x}{\delta(x)\Delta(x)}\,\tilde{D}_{10}^+(x)\left[\varpi + \frac{x}{(1-x)\Delta(x)}\,\tilde{N}_0^+(x)\right] \\
& + \left[1 - \frac{\varpi x}{\delta(x)\Delta(x)}\,\tilde{D}_{11}^+(x)\right]\left[\varpi + \frac{x}{(1-x)\Delta(x)}\,\tilde{N}_1^+(x)\right] \\
& - \frac{\varpi y}{\delta(y)\Delta(y)}\,\tilde{D}_{10}^-(y)\left[\varpi + \frac{y}{(1-y)\Delta(y)}\,\tilde{N}_0^-(y)\right] \\
& + \left[1 - \frac{\varpi y}{\delta(y)\Delta(y)}\,\tilde{D}_{11}^-(y)\right]\left[\varpi + \frac{y}{(1-y)\Delta(y)}\,\tilde{N}_1^-(y)\right] - \varpi.
\end{aligned}$$

□

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